

# A HIERARCHY OF PARAMETRIZING VARIETIES FOR REPRESENTATIONS

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*Dedicated to Carl Faith and Barbara Osofsky*

**ABSTRACT.** The primary purpose is to introduce and explore projective varieties,  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ , parametrizing the full collection of those modules over a finite dimensional algebra  $\Lambda$  which have dimension vector  $\mathbf{d}$ . These varieties extend the smaller varieties previously studied by the author; namely, the projective varieties encoding those modules with dimension vector  $\mathbf{d}$  which, in addition, have a preassigned top or radical layering. Each of the  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  is again partitioned by the action of a linear algebraic group, and covered by certain representation-theoretically defined affine subvarieties which are stable under the unipotent radical of the acting group. A special case of the pertinent theorem served as a cornerstone in the work on generic representations by Babson, Thomas, and the author. Moreover, applications are given to the study of degenerations.

## 1. INTRODUCTION AND NOTATION

Our primary aim is to extend some of the concepts, constructions and results from [8], [9], and [1] for wider applicability towards exploring the representation theory of a basic finite dimensional algebra  $\Lambda$  over an algebraically closed field  $K$ . These varieties constitute the foundation for present and future work on degenerations, irreducible components and generic representations. More specifically, our purpose is to introduce a projective parametrizing variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  for the full collection of isomorphism classes of modules with a given dimension vector  $\mathbf{d} = (d_1, \dots, d_n)$  recording the composition multiplicities of the simple objects  $S_1, \dots, S_n$  in  $\Lambda\text{-mod}$ . This large variety is to supplement the projective variety  $\mathfrak{Grass}_{\mathbf{d}}^T$  and its subvariety  $\mathfrak{Grass}(\mathbb{S})$ , which encode the  $\mathbf{d}$ -dimensional modules  $M$  with fixed top  $T = M/JM$  and fixed radical layering  $\mathbb{S} = (J^l M/J^{l+1} M)_{0 \leq l \leq L}$ , respectively [8, 9, 1]; here  $J$  denotes the Jacobson radical of  $\Lambda$  and  $L + 1$  its nilpotency index. This permits us to study “global” problems by moving back and forth between local and global settings. (“GRASS” and “ $\mathfrak{Grass}$ ” both stand for “Grassmann”, indicating that the varieties carrying one of these labels are subvarieties of Grassmannians.)

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The new variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  and its subvarieties  $\text{GRASS}_{\mathbf{d}}^T$  and  $\text{GRASS}(\mathbb{S})$  – the latter parametrizing the same isomorphism classes of modules as  $\mathbf{Grass}_{\mathbf{d}}^T$  and  $\mathbf{Grass}(\mathbb{S})$ , respectively – again come equipped with an algebraic group action; as in all of the previously considered cases, the orbits partition the given variety into strata representing the different isomorphism classes of modules. The acting group in the new setting is in turn larger than the one that accompanies the smaller varieties  $\mathbf{Grass}_{\mathbf{d}}^T$  and  $\mathbf{Grass}(\mathbb{S})$ , whence the enlarged projective setting  $\text{GRASS}$  loses the advantage of comparatively small orbit dimensions. However, it retains several crucial assets, notably completeness. An additional plus of all the varieties arising in the Grassmannian scenario lies in the fact that the acting group typically has a large unipotent radical; this brings a cache of methods to bear which supplement those available for the reductive group actions in the affine scenario. Yet, more importantly, the Grassmannian varieties  $\text{GRASS}(\mathbb{S})$  and  $\mathbf{Grass}(\mathbb{S})$  are covered by certain representation-theoretically defined open subvarieties –  $\text{GRASS}(\sigma)$  and  $\mathbf{Grass}(\sigma)$  respectively – which are particularly accessible: Namely, the modules with fixed radical layering  $\mathbb{S}$  are further subdivided into classes of modules characterized by much finer structural invariants, their “skeleton”  $\sigma$  (see Definition 3.1). On the level of the  $\mathbf{Grass}(\sigma)$ , representation-theoretic investigations can be carried out more effectively than on the levels located above, with good venues for moving information back “up the ladder”.

To benefit from the increased representation-theoretic transparency of the variety  $\mathbf{Grass}_{\mathbf{d}}^T$  representing the same objects as  $\text{GRASS}_{\mathbf{d}}^T$ , the practical philosophy is as follows: When a global problem tackled in the large variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  has been played down to  $\text{GRASS}_{\mathbf{d}}^T$  (non-closed and hence non-projective), consisting of the points encoding modules with a fixed top  $T$ , it is advantageous to bring the smaller projective counterpart  $\mathbf{Grass}_{\mathbf{d}}^T$  into the picture. The governing idea is to reduce any given problem to the smallest possible setting to render it more tractable and then move in reverse. So, in other words, it is advantageous to explore both the large and small scenarios side by side.

In order to place the large varieties  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  into context, we begin by reminding the reader of the other candidates on the previously defined list of nested “small” parametrizing varieties of representations:

$$\mathbf{Grass}_{\mathbf{d}}^T \supseteq \mathbf{Grass}_{\mathbf{d}}^T \supseteq \mathbf{Grass}(\mathbb{S}) \supseteq \mathbf{Grass}(\sigma).$$

Here  $d$  is the total dimension of  $\mathbf{d}$ , and  $\mathbf{Grass}_{\mathbf{d}}^T$  parametrizes the modules with dimension  $d$  and top  $T$ . By  $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$  we denote a sequence of semisimple modules with “top”  $\mathbb{S}_0 = T$  such that  $\mathbf{d} = \underline{\dim} \bigoplus_{0 \leq l \leq L} \mathbb{S}_l$ ; it represents a typical radical layering of a module with top  $T$  and dimension vector  $\mathbf{d}$ . In order to descend an additional step to the varieties labeled  $\mathbf{Grass}(\sigma)$ , we present  $\Lambda$  as a path algebra modulo relations, that is, in the form  $\Lambda = K\Gamma/I$ , where  $\Gamma$  is the Gabriel quiver of  $\Lambda$ , and  $I$  an admissible ideal in the path algebra  $K\Gamma$ . This allows us to organize  $\Lambda$ -modules in terms of their “skeleton”, roughly speaking, path bases of  $\Lambda$ -modules which reflect their underlying  $K\Gamma$ -module structure. Starting with a skeleton  $\sigma$  compatible with the sequence  $\mathbb{S}$  (Definition 3.5), we denote by  $\mathbf{Grass}(\sigma)$  the subvariety of  $\mathbf{Grass}(\mathbb{S})$  which consists of the points corresponding to modules with skeleton  $\sigma$ .

We will keep the old notation for compatibility with prior work, and distinguish the larger varieties to be defined and discussed in the sequel by capital letters in order to place emphasis on the big versus the small scenario. The enlarged setting consists of

$$\text{GRASS}_d(\Lambda) \supseteq \text{GRASS}_{\mathbf{d}}(\Lambda) \supseteq \text{GRASS}_{\mathbf{d}}^T \supseteq \text{GRASS}(\mathbb{S}) \supset \text{GRASS}(\sigma),$$

where  $\text{GRASS}_d(\Lambda)$  and  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  are projective varieties representing all modules with dimension  $d$ , resp. dimension vector  $\mathbf{d}$ , and  $\text{GRASS}_{\mathbf{d}}^T$ ,  $\text{GRASS}(\mathbb{S})$ ,  $\text{GRASS}(\sigma)$  are the locally closed subvarieties consisting of the points which, in analogy with the previous list, correspond to the modules with top  $T$ , radical layering  $\mathbb{S}$ , and skeleton  $\sigma$ , respectively.

We preview the main results of the paper, Theorems 3.12 and 3.17, in a loosely phrased version. The “small version” has already been used without proof, with reference to the present article, in [1] and [9]; in both of these investigations, it plays a pivotal role. Moreover, it served as basis for an algorithm, developed by Babson, Thomas and the author [2], which computes polynomials for the smallest of the considered varieties – that is, for the  $\mathfrak{Grass}(\sigma)$  – from the quiver  $\Gamma$  and generators for the ideal  $I$ .

Suppose  $P$  and  $\mathbf{P}$  are projective covers of  $T$  and  $\bigoplus_{1 \leq i \leq n} S_i^{d_i}$ , respectively. Then  $\mathfrak{Grass}_{\mathbf{d}}^T$  and  $\text{GRASS}_{\mathbf{d}}^T$  consist of the submodules  $C$  of  $JP$ , resp.  $\mathbf{P}$ , with the property that the factor module  $P/C$ , resp.  $\mathbf{P}/C$ , has top  $T$  and dimension vector  $\mathbf{d}$ . Denote by  $\phi$ , resp.  $\Phi$ , the map sending  $C$  to the isomorphism class of  $P/C$ , resp.  $\mathbf{P}/C$ .

**Theorem.** *Let  $\sigma$  be a skeleton of dimension vector  $\mathbf{d}$  which is compatible with  $\mathbb{S}$ . Then the set  $\mathfrak{Grass}(\sigma)$  consisting of the points in  $\mathfrak{Grass}_{\mathbf{d}}^T$  which correspond to modules with skeleton  $\sigma$  is an open subvariety of  $\mathfrak{Grass}(\mathbb{S})$ ; analogously,  $\text{GRASS}(\sigma)$  is an open subvariety of  $\text{GRASS}(\mathbb{S})$ .*

*There is an isomorphism  $\psi$  from  $\mathfrak{Grass}(\sigma)$  onto a closed subvariety of an affine space  $\mathbb{A}^N$ , together with an explicitly specified map  $\chi$  from  $\text{Im}(\psi)$  onto the set of isomorphism classes of modules with skeleton  $\sigma$ , such that the following triangle commutes:*

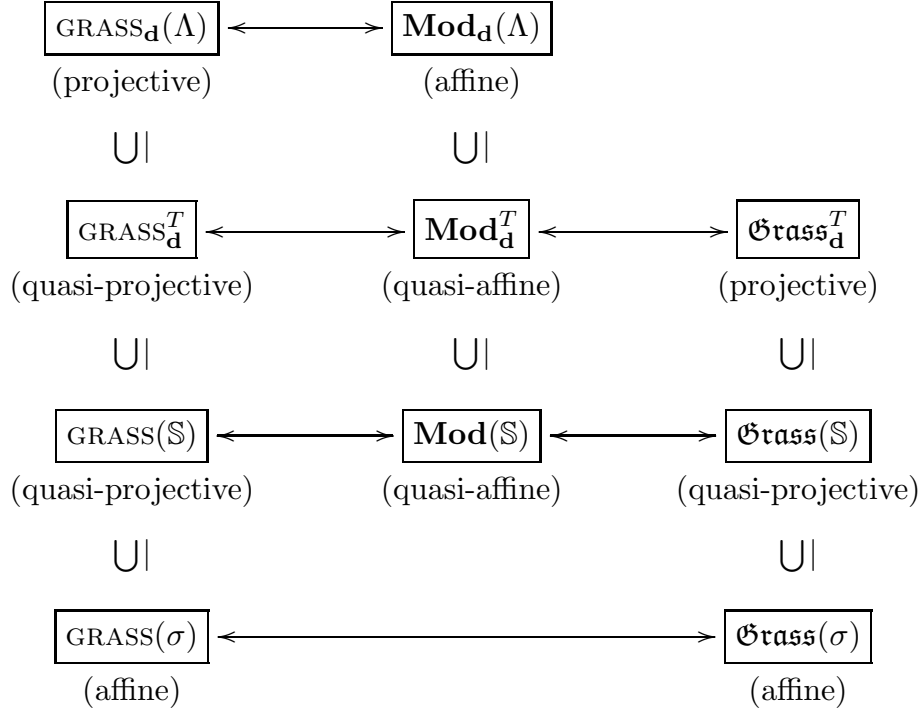
$$\begin{array}{ccc} \mathbb{A}^N \supseteq \text{Im}(\psi) & & \\ \downarrow \psi^{-1} & \searrow \chi & \\ \mathfrak{Grass}_{\mathbf{d}}^T \supseteq \mathfrak{Grass}(\sigma) & \nearrow \phi & \{\text{isom. types of } \Lambda\text{-modules with skeleton } \sigma\} \end{array}$$

*Polynomial equations for the isomorphic copy  $\text{Im}(\psi)$  of  $\mathfrak{Grass}(\sigma)$  in  $\mathbb{A}^N$  can be algorithmically obtained from  $\Gamma$  and left ideal generators for  $I$ .*

*In the big scenario, there exist an isomorphism  $\Psi$  from  $\text{GRASS}(\sigma)$  to a closed subset of an affine space and a map  $\chi$  from  $\text{Im}(\Psi)$  to the set of isomorphism classes of  $\Lambda$ -modules with skeleton  $\sigma$ , giving rise to a corresponding commutative diagram.*

Moreover,  $\text{GRASS}(\sigma) \cong \mathfrak{Grass}(\sigma) \times \mathbb{A}^{N_0}$ , where  $N_0$  is determined by the semisimple sequence  $\mathbb{S}$  alone.

We follow with a schematic overview of the relevant varieties. Those in the first three slots of the left column below carry a canonical action by the big automorphism group  $\text{Aut}_\Lambda(\mathbf{P})$ , those in the corresponding slots of the central column are equipped with the conjugation action by  $\text{GL}_d$ , and the first two varieties in the rightmost column with an action by  $\text{Aut}_\Lambda(P)$  that parallels that of  $\text{Aut}_\Lambda(\mathbf{P})$ . Moreover, the horizontal double arrows indicate easy transfer of information. The varieties on the same level parametrize the same set of isomorphism types of modules, with geometric information being transferrable between the two sides via Propositions 2.1 and 2.5. In the first three rows, this concerns information about action-stable subsets. The affine varieties in the last row are not stable under  $\text{Aut}_\Lambda(\mathbf{P})$  or  $\text{Aut}_\Lambda(P)$  in general, but only stable under the action of the unipotent radical of  $\text{Aut}_\Lambda(\mathbf{P})$ , resp. of  $\text{Aut}_\Lambda(P)$ , and a maximal torus in the relevant automorphism group; on this lowest level, the transfer of information is by way of the final statement of the above theorem.



In the final section, we present some first applications of the new Grassmannians to degenerations.

Throughout,  $\Lambda$  denotes a basic finite dimensional algebra over an algebraically closed field  $K$ . Hence, we do not lose generality in assuming that  $\Lambda = K\Gamma/I$  is a path algebra modulo relations as above. The vertices  $e_1, \dots, e_n$  of  $\Gamma$  will be identified with the primitive idempotents of  $\Lambda$  corresponding to the paths of length zero. As is well-known, the left ideals  $\Lambda e_i$  then represent all indecomposable projective (left)  $\Lambda$ -modules, up to isomorphism, and

the factors  $S_i = \Lambda e_i / J e_i$ , where  $J$  is the Jacobson radical of  $\Lambda$ , form a set of representatives for the simple left  $\Lambda$ -modules. By  $L + 1$  we will denote the Loewy length of  $\Lambda$ . Moreover, we will observe the following conventions: The product  $pq$  of two paths  $p$  and  $q$  in  $K\Gamma$  stands for “first  $q$ , then  $p$ ”; in particular,  $pq$  is zero unless the end point of  $q$  coincides with the starting point of  $p$ . In accordance with this convention, we call a path  $p_1$  an *initial subpath* of  $p$  if  $p = p_2 p_1$  for some path  $p_2$ . A *path in  $\Lambda$*  is a residue class of the form  $p + I$ , where  $p$  is a path in  $KQ \setminus I$ ; we will suppress the residue notation, provided there is no risk of ambiguity. Further, we will gloss over the distinction between the left  $\Lambda$ -structure of a module  $M \in \Lambda\text{-Mod}$  and its induced  $K\Gamma$ -module structure when there is no danger of confusion. An element  $x$  of  $M$  will be called a *top element* of  $M$  if  $x \notin JM$  and  $x$  is *normed* by some  $e_i$ , meaning that  $x = e_i x$ . Any collection  $x_1, \dots, x_m$  of top elements of  $M$  generating  $M$  and linearly independent modulo  $JM$  will be referred to as a *full sequence of top elements in  $M$* .

## 2. OLD AND NEW PARAMETRIZING VARIETIES FOR REPRESENTATIONS

We begin by laying out the various varieties listed in the three upper rows of the above diagram. In subsection 2.A, we briefly remind the reader of the classical affine parametrizing varieties  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ . In 2.B, we review crucial properties of the subvariety  $\mathbf{Mod}_{\mathbf{d}}^T$  and its projective counterpart  $\mathfrak{Grass}_{\mathbf{d}}^T$ . In 2.C, we cut down further in size to  $\mathbf{Mod}(\mathbb{S})$  and  $\mathfrak{Grass}(\mathbb{S})$ . Next, in 2.D, we introduce the larger projective variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  and its subvarieties  $\text{GRASS}_{\mathbf{d}}^T$  and  $\text{GRASS}(\mathbb{S})$ , and compare the “big” and “small” settings following the horizontal double arrows. In 2.E, we use the sets  $\mathbf{Mod}(\mathbb{S})$  and  $\text{GRASS}(\mathbb{S})$  to build useful closed subvarieties of  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  and  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ . In 2.F, finally, we discuss the structure of the automorphism groups acting on the projective parametrizing varieties.

### 2.A. Reminder: The classical affine variety $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ .

Let  $a_1, \dots, a_r$  be a set of algebra generators for  $\Lambda$  over  $K$ . A convenient set of such generators consists of the primitive idempotents (= vertices)  $e_1, \dots, e_n$  together with the (residue classes in  $\Lambda$  of the) arrows in  $\Gamma$ . Recall that, for any natural number  $d$ , the classical affine variety of  $d$ -dimensional representations of  $\Lambda$  can be described in the form

$$\mathbf{Mod}_d(\Lambda) = \left\{ (x_i) \in \prod_{1 \leq i \leq r} \text{End}_K(K^d) \mid \text{the } x_i \text{ satisfy all relations satisfied by the } a_i \right\}.$$

As is well-known, the isomorphism classes of  $d$ -dimensional (left)  $\Lambda$ -modules are in one-to-one correspondence with the orbits of  $\mathbf{Mod}_d(\Lambda)$  under the  $\text{GL}_d$ -conjugation action. Moreover, the connected components of  $\mathbf{Mod}_d(\Lambda)$  are known to be in natural one-to-one correspondence with the dimension vectors  $\mathbf{d} = (d_1, \dots, d_n)$  of total dimension  $d$ , meaning  $|\mathbf{d}| = \sum_i d_i = d$ . Namely, the connected component corresponding to  $\mathbf{d}$  is

$$\mathbf{Mod}_{\mathbf{d}}(\Lambda) = \{x \in \mathbf{Mod}_d(\Lambda) \mid \text{the module corresponding to } x \text{ has dimension vector } \mathbf{d}\};$$

see [6], Corollary 1.4. In particular,  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  is a closed subvariety of  $\mathbf{Mod}_d(\Lambda)$ , and hence again affine. In the following, we will exclusively focus on varieties parametrizing modules with fixed dimension vector  $\mathbf{d}$  (as opposed to fixed dimension  $d$ ), since that will not result in any restrictions to applicability.

If  $I = 0$ , that is, if  $\Lambda$  is hereditary, the connected varieties  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  are even irreducible, but this fails already in small non-hereditary examples. For instance, when  $\mathbf{d} = (1, 1)$  and  $\Lambda = K\Gamma/I$ , where  $\Gamma$  is the quiver  $1 \rightleftarrows 2$  and  $I$  is generated by the paths of length 2, the variety  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ , and analogously  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ , has two irreducible components, namely the orbit closures of the two uniserial modules of dimension 2 (see Examples 2.8(1) for more detail in the Grassmannian setting).

## 2.B. The quasi-affine variety $\mathbf{Mod}_{\mathbf{d}}^T$ and its counterpart, the projective variety $\text{Grass}_{\mathbf{d}}^T$ .

The two isomorphism invariants of a  $\Lambda$ -module  $M$  which will be pivotal here are the *top* and the *radical layering* of  $M$ , the latter being a refinement of the former. The *top* of  $M$  is defined as  $M/JM$ , and the *radical layering* as the sequence  $\mathbb{S}(M) = (J^l M / J^{l+1} M)_{0 \leq l \leq L}$  of semisimple modules. We will say that a module  $M$  has *top*  $T$  in case  $M/JM \cong T$ . In fact, we will identify isomorphic semisimple modules, and thus write  $M/JM = T$  in this situation. In light of our identification, the set of all finite dimensional semisimple modules is partially ordered under inclusion. Specifically, we write  $T \leq T'$  to denote that a semisimple module  $T$  is (isomorphic to) a submodule of a semisimple module  $T'$ .

*The following choices and notation will be observed throughout:* We fix a semisimple  $\Lambda$ -module  $T$ , say

$$T = \bigoplus_{1 \leq i \leq n} S_i^{t_i},$$

with dimension vector  $\mathbf{t} = (t_1, \dots, t_n)$  and total dimension  $t = \sum_i t_i$ , and denote by

$$P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$$

the *distinguished projective cover* of  $T$ ; this means that  $\Lambda z_1, \dots, \Lambda z_{t_1}$  are isomorphic to  $\Lambda e_1$ , while  $\Lambda z_{t_1+1}, \dots, \Lambda z_{t_1+t_2}$  are isomorphic to  $\Lambda e_2$ , and so on. Here each  $z_r$  is a top element of  $P$  normed by a primitive idempotent  $e(r)$ , that is,  $z_r = e(r)z_r$ , and we refer to  $z_1, \dots, z_t$  as the *distinguished sequence of top elements* of  $P$ . Clearly,  $P$  is a projective cover of any module with top  $T$ ; in other words, the modules with top  $T$  are precisely the quotients  $P/C$  with  $C \subseteq JP$ , up to isomorphism. Finally, we fix a dimension vector  $\mathbf{d} = (d_1, \dots, d_n)$  of total dimension  $d$  such that  $\mathbf{t} \leq \mathbf{d}$  in the componentwise partial order on  $\mathbb{N}_0^n$ .

The tops lead to a first rough subdivision of  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ : By  $\mathbf{Mod}_{\mathbf{d}}^T$  we denote the locally closed subvariety of  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  which consists of the points representing the modules with top  $T$ , that is,

$$\mathbf{Mod}_{\mathbf{d}}^T = \{x \in \mathbf{Mod}_{\mathbf{d}}(\Lambda) \mid \text{the module corresponding to } x \text{ has top } T\}.$$

While the varieties  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  are connected, the subvarieties  $\mathbf{Mod}_{\mathbf{d}}^T$  need not be.

As for the projective counterpart of  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ : In [8], we defined  $\mathbf{Grass}_d^T$  to be the following closed subvariety of the classical Grassmannian  $\mathfrak{Gr}(\dim P - d, JP)$  of  $(\dim P - d)$ -dimensional subspaces of the  $K$ -space  $JP$ , namely,

$$\mathbf{Grass}_d^T = \{C \in \mathfrak{Gr}(\dim P - d, JP) \mid C \text{ is a } \Lambda\text{-submodule of } JP\}.$$

This variety comes with an obvious surjection

$$\mathbf{Grass}_d^T \longrightarrow \{\text{isomorphism classes of } d\text{-dimensional } \Lambda\text{-modules with top } T\},$$

sending  $C$  to the class of  $P/C$ . Clearly, the fibres of this map coincide with the orbits of the natural  $\text{Aut}_{\Lambda}(P)$ -action on  $\mathbf{Grass}_d^T$ .

Here, we will consistently keep the dimension vector  $\mathbf{d}$  (not only the total dimension  $d$ ) of the considered representations fixed. The projective counterpart to the quasi-affine variety  $\mathbf{Mod}_{\mathbf{d}}^T$  is a closed (and hence again projective) subvariety  $\mathbf{Grass}_{\mathbf{d}}^T$  of  $\mathbf{Grass}_d^T$ . Namely,

$$\mathbf{Grass}_{\mathbf{d}}^T = \{C \in \mathbf{Grass}_d^T \mid \underline{\dim} P/C = \mathbf{d}\}.$$

Clearly,  $\mathbf{Grass}_{\mathbf{d}}^T$  is stable under the  $\text{Aut}_{\Lambda}(P)$ -action, and the corresponding orbits are in one-to-one correspondence with the isomorphism classes of modules with top  $T$  and dimension vector  $\mathbf{d}$ .

We note that  $\mathbf{Grass}_{\mathbf{d}}^T$  is irreducible (or connected) precisely when  $\mathbf{Mod}_{\mathbf{d}}^T$  has this property. This follows from the following proposition due to Bongartz and the author, which links the “relative geometry” of the  $\text{GL}_d$ -stable subsets of  $\mathbf{Mod}_{\mathbf{d}}^T$  to that of the  $\text{Aut}_{\Lambda}(P)$ -stable subsets of  $\mathbf{Grass}_{\mathbf{d}}^T$ .

**Proposition 2.1.** (See [4], Proposition C.) *The assignment  $\text{Aut}_{\Lambda}(P).C \mapsto \text{GL}_d.x$ , which pairs orbits  $\text{Aut}_{\Lambda}(P).C \subseteq \mathbf{Grass}_{\mathbf{d}}^T$  and  $\text{GL}_d.x \subseteq \mathbf{Mod}_{\mathbf{d}}^T$  representing the same  $\Lambda$ -module up to isomorphism, induces an inclusion-preserving bijection*

$$\Phi : \{\text{Aut}_{\Lambda}(P)\text{-stable subsets of } \mathbf{Grass}_{\mathbf{d}}^T\} \rightarrow \{\text{GL}_d\text{-stable subsets of } \mathbf{Mod}_{\mathbf{d}}^T\}$$

*which preserves and reflects openness, closures, connectedness, irreducibility, and types of singularities.*  $\square$

In particular, this correspondence permits transfer of information concerning the irreducible components of any locally closed  $\text{GL}_d$ -stable subvariety of  $\mathbf{Mod}_{\mathbf{d}}^T$  to the irreducible components of the corresponding  $\text{Aut}_{\Lambda}(P)$ -stable subvariety of  $\mathbf{Grass}_{\mathbf{d}}^T$ , and vice versa. Indeed, given that the acting groups,  $\text{GL}_d$  and  $\text{Aut}_{\Lambda}(P)$ , are connected, all of their orbits are irreducible. Therefore all such irreducible components are again stable under the respective actions, meaning that  $\Phi$  restricts to a bijection between the collection of irreducible components of  $\mathbf{Grass}_{\mathbf{d}}^T$  on one hand, and that of  $\mathbf{Mod}_{\mathbf{d}}^T$  on the other. Similarly, information regarding degenerations can be shifted back and forth across the bridge  $\Phi$  (see Section 4).

### 2.C. The subvarieties $\mathbf{Mod}(\mathbb{S}) \subseteq \mathbf{Mod}_{\mathbf{d}}^T$ and $\mathfrak{Grass}(\mathbb{S}) \subseteq \mathfrak{Grass}_{\mathbf{d}}^T$ .

The radical layering of modules provides us with a further partition of  $\mathbf{Mod}_{\mathbf{d}}^T$  and  $\mathfrak{Grass}_{\mathbf{d}}^T$  into pairwise disjoint locally closed subvarieties. (In general, this partition fails to be a stratification in the technical sense, however, even when  $\Lambda$  is hereditary.)

**Definition 2.2 and first comments.** A *semisimple sequence* with dimension vector  $\mathbf{d}$  and top  $T$  is any sequence  $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$  of semisimple modules such that  $\mathbb{S}_0 = T$  and the semisimple module  $\sum_{0 \leq l \leq L} \mathbb{S}_l$  has dimension vector  $\mathbf{d}$ . Moreover (in order to rule out uninteresting sequences), we require that each  $\mathbb{S}_l \leq J^l P / J^{l+1} P$  for  $0 \leq l \leq L$ . (This condition is intrinsic to  $\mathbb{S}$ , since  $P$  is the projective cover of  $\mathbb{S}_0$ .) Since we identify semisimple modules with their isomorphism classes, a semisimple sequence amounts to a matrix of discrete invariants keeping count of the multiplicities of the simple modules in the individual slots of  $\mathbb{S}$ . The radical layering of any finitely generated  $\Lambda$ -module  $M$  yields a semisimple sequence, namely  $(M/JM, JM/J^2M, \dots, J^L M)$ . As noted in 2.B, we denote this sequence by  $\mathbb{S}(M)$ .

Given such a semisimple sequence  $\mathbb{S}$ , we define the following action-stable locally closed subvarieties of  $\mathbf{Mod}_{\mathbf{d}}^T$  and  $\mathfrak{Grass}_{\mathbf{d}}^T$ , respectively:

$$\begin{aligned} \mathbf{Mod}(\mathbb{S}) &= \{x \in \mathbf{Mod}_{\mathbf{d}}^T \mid x \text{ corresponds to a module with radical layering } \mathbb{S}\} \\ \mathfrak{Grass}(\mathbb{S}) &= \{C \in \mathfrak{Grass}_{\mathbf{d}}^T \mid \mathbb{S}(P/C) = \mathbb{S}\}. \end{aligned}$$

Clearly, the one-to-one correspondence  $\Phi$  of Proposition 2.1, between the  $\text{Aut}_{\Lambda}(P)$ -stable subsets of  $\mathfrak{Grass}_{\mathbf{d}}^T$  and the  $\text{GL}_d$ -stable subsets of  $\mathbf{Mod}_{\mathbf{d}}^T$ , restricts to a correspondence between the  $\text{Aut}_{\Lambda}(P)$ -stable subsets of  $\mathfrak{Grass}(\mathbb{S})$  and the  $\text{GL}_d$ -stable subsets of  $\mathbf{Mod}(\mathbb{S})$ , which respects geometric properties in the sense of the proposition. In particular, irreducible components are preserved by  $\Phi$  and  $\Phi^{-1}$ .

### 2.D. The projective variety $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

We next embed the projective varieties  $\mathfrak{Grass}_{\mathbf{d}}^T$  for a fixed dimension vector  $\mathbf{d}$  and variable tops  $T$  (with dimension vectors  $\mathbf{t} \leq \mathbf{d}$ ) into a bigger variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  – still projective – in order to obtain a projective counterpart to the full classical affine variety  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ . The projective setting still offers advantages supplementing those of the classical affine one, due to compactness and to the fact that it invites methodological alternatives; indeed the different structures of the operating groups – reductive in the classical scenario, containing a big unipotent radical in the Grassmannian setting – bring different lines of the existing theory of algebraic group actions to bear. However, the asset of comparative smallness of  $\mathfrak{Grass}_{\mathbf{d}}^T$  and its orbits in comparison with the classical setting is lost. In fact, in general, we will find infinitely many copies of  $\mathfrak{Grass}_{\mathbf{d}}^T$  inside  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

Let  $Q$  be any finitely generated projective  $\Lambda$ -module. When working with such a  $Q$ , in place of the projective cover  $P$  of  $T$ , we transfer the notation  $z_r$  and  $e(r)$  to  $Q$ . That is, we specify a distinguished sequence of top elements  $z_1, \dots, z_m$  in  $Q$ , denote the primitive idempotent norming  $z_r$  by  $e(r)$ , and refer to  $Q = \bigoplus_{1 \leq r \leq m} \Lambda z_r$  as the distinguished projective cover of the semisimple module  $Q/JQ$ . Of particular interest will be the case



where  $Q/JQ = \bigoplus_{1 \leq r \leq n} S_i^{d_i}$  is the semisimple module with dimension vector  $\mathbf{d}$ . In that case, we will denote the distinguished projective cover  $Q$  by  $\mathbf{P}$ .

Any  $\Lambda$ -module with  $\text{top} \leq Q/JQ$  is a quotient of  $Q$ , although  $Q$  need not be its projective cover. To accomodate this fact, we simply modify the definition of  $\mathfrak{Grass}_{\mathbf{d}}^T$  to allow submodules of  $Q$  that might not be contained in  $JQ$ .

**Definition 2.3 and comments.** The subset

$$[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q := \{C \in \mathfrak{Gr}(\dim Q - d, Q) \mid C \text{ is a } \Lambda\text{-submodule of } Q \text{ with } \underline{\dim} Q/C = \mathbf{d}\}$$

is a closed, and hence projective, subvariety of the classical Grassmann variety  $\mathfrak{Gr}(\dim Q - d, Q)$ . Clearly, it parametrizes all isomorphism classes of modules with dimension vector  $\mathbf{d}$  and  $\text{top} \leq Q/JQ$  via the map that sends  $C$  to the class of  $Q/C$ . Again, we let the automorphism group of the underlying projective module, that is,  $\text{Aut}_{\Lambda}(Q)$  in the present setting, act canonically on  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ . As we will see in Observation 2.4 below, the isomorphism classes of modules parametrized here are in bijective correspondence with the  $\text{Aut}_{\Lambda}(Q)$ -orbits of  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ .

In the most frequently considered special case, where  $Q = \mathbf{P}$ , we denote  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$  by  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ . So, explicitly,  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  is the following projective variety:

$$\text{GRASS}_{\mathbf{d}}(\Lambda) = \{C \in \mathfrak{Gr}(\dim \mathbf{P} - d, \mathbf{P}) \mid C \text{ is a } \Lambda\text{-submodule of } \mathbf{P} \text{ with } \underline{\dim} \mathbf{P}/C = \mathbf{d}\}.$$

In this latter case, the assignment  $C \mapsto \mathbf{P}/C$  yields a parametrization of all modules with dimension vector  $\mathbf{d}$ , up to isomorphism, irrespective of their tops, and the fibers of this parametrization coincide with the  $\text{Aut}_{\Lambda}(\mathbf{P})$ -orbits of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

**Observation 2.4.** *Let  $Q$  be a finitely generated projective  $\Lambda$ -module as introduced ahead of Definition 2.3. The isomorphism classes of left  $\Lambda$ -modules with dimension vector  $\mathbf{d}$  and  $\text{top} \leq Q/JQ$  are in one-to-one correspondence with the  $\text{Aut}_{\Lambda}(Q)$ -orbits of  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ , where  $\text{Aut}_{\Lambda}(Q).C \subseteq [\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$  corresponds to the isomorphism class of  $Q/C$ .*

*Proof.* Clearly, every module with dimension vector  $\mathbf{d}$  and  $\text{top}$  contained in  $Q/JQ$  is of the form  $Q/C$  for a suitable point  $C \in [\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ , and, given  $f \in \text{Aut}_{\Lambda}(Q)$ , we have  $Q/(f.C) \cong Q/C$ .

Suppose that  $D$  is a point in  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$  such that  $Q/D \cong Q/C$ , via some isomorphism  $f : Q/C \rightarrow Q/D$  say. Denote by  $\text{can}_C$  and  $\text{can}_D$  the canonical epimorphisms  $Q \rightarrow Q/C$  and  $Q \rightarrow Q/D$ , respectively. Moreover, let  $\pi_C : Q' \rightarrow Q/C$  be a projective cover of  $Q/C$ , and  $\iota_C : Q' \rightarrow Q$  a (necessarily split) embedding of  $Q'$  into  $Q$  such that the composition  $(\text{can}_C) \circ (\iota_C) = \pi_C$ . Then  $Q = \iota_C(Q') \oplus C'$  for a suitable direct summand  $C' \subseteq C$ , and  $C = \text{Ker}(\pi_C \iota_C^*) \oplus C'$ , where  $\iota_C^* : \iota_C(Q') \rightarrow Q'$  is inverse to  $\iota_C$ . Since  $Q/D \cong Q/C$ , there is also a projective cover  $\pi_D : Q' \rightarrow Q/D$ , and symmetrically, we obtain a split embedding  $\iota_D : Q' \rightarrow Q$  and a direct summand  $D'$  of  $D$  such that all of the above

conditions are satisfied on replacement of  $C$  by  $D$ . In particular,  $Q \cong Q' \oplus C' \cong Q' \oplus D'$ . Consequently,  $C' \cong D'$ , via some isomorphism  $h' : C' \rightarrow D'$  say.

The maps  $\pi_C \iota_C^*$  and  $\pi_D \iota_D^*$  are projective covers of  $Q/C$  and  $Q/D$ , respectively, whence the isomorphism  $f$  lifts to an isomorphism  $h''$  making the following diagram commute:

$$\begin{array}{ccc} \iota_C(Q') & \xrightarrow{h''} & \iota_D(Q') \\ \pi_C \iota_C^* \downarrow & & \downarrow \pi_D \iota_D^* \\ Q/C & \xrightarrow{f} & Q/D \end{array}$$

Then  $h''(\text{Ker}(\pi_C \iota_C^*)) = \text{Ker}(\pi_D \iota_D^*)$ , and we deduce that  $h = h' \oplus h''$  is an automorphism of  $Q$  which takes  $C$  to  $D$ . In other words,  $D$  lies in the  $\text{Aut}_\Lambda(Q)$ -orbit of  $C$ , as required.  $\square$

So, in particular, the  $\text{Aut}_\Lambda(\mathbf{P})$ -orbits of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  are in natural bijective correspondence with the isomorphism classes of left  $\Lambda$ -modules having dimension vector  $\mathbf{d}$ . We record an analogue of Proposition 2.1 for the enlarged setting. For a proof, we again refer to [4], Proposition C.

**Proposition 2.5.** (twin of 2.1) *The assignment  $\text{Aut}_\Lambda(\mathbf{P}).C \mapsto \text{GL}_d.x$ , which pairs orbits  $\text{Aut}_\Lambda(\mathbf{P}).C \subseteq \text{GRASS}_{\mathbf{d}}(\Lambda)$  and  $\text{GL}_d.x \subseteq \mathbf{Mod}_{\mathbf{d}}(\Lambda)$  representing the same  $\Lambda$ -module up to isomorphism, induces an inclusion-preserving bijection*

$$\Phi : \{\text{Aut}_\Lambda(\mathbf{P})\text{-stable subsets of } \text{GRASS}_{\mathbf{d}}(\Lambda)\} \rightarrow \{\text{GL}_d\text{-stable subsets of } \mathbf{Mod}_{\mathbf{d}}(\Lambda)\}$$

*which preserves and reflects openness, closures, connectedness, irreducibility, and types of singularities.*  $\square$

**Remark 2.6.** In analogy with the above constructions, one may define a projective variety  $\text{GRASS}_d(\Lambda)$  encoding all of the  $d$ -dimensional  $\Lambda$ -modules. Namely, let  $Q$  be the free left  $\Lambda$ -module  $\Lambda^{nd}$ , and set

$$\text{GRASS}_d(\Lambda) := \{C \in \mathfrak{Gr}(\dim Q - d, Q) \mid C \text{ is a } \Lambda\text{-submodule of } Q\}.$$

This variety is, in turn, equipped with a natural  $\text{Aut}_\Lambda(Q)$ -action, the orbits of which parametrize the isomorphism classes of the left  $\Lambda$ -modules of dimension  $d$ .

As in Proposition 2.5, we then obtain a map from the action-stable subsets of  $\text{GRASS}_d(\Lambda)$  to the action-stable subsets of  $\mathbf{Mod}_d(\Lambda)$ , which preserves and reflects relative geometric properties. Combining this observation with the well-known fact that the connected components of  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  coincide with the subvarieties  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ , we obtain that the subsets

$$\{C \in \text{GRASS}_d(\Lambda) \mid Q/C \text{ has dimension vector } \mathbf{d}\},$$

where  $\mathbf{d}$  traces the dimension vectors of total dimension  $d$ , are the connected components of  $\text{GRASS}_d(\Lambda)$ . In particular, they are closed subvarieties of  $\text{GRASS}_d(\Lambda)$ .

In light of Remark 2.6, it will not impact the usefulness of the considered varieties to restrict to  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  and  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ .

Fix a semisimple module  $T$  as specified at the beginning of Section 2.B, and suppose  $T \leq \mathbf{P}/J\mathbf{P}$ , where, as above,  $\mathbf{P} = \bigoplus_{1 \leq r \leq d} \Lambda z_r$  is the distinguished projective cover of  $\bigoplus_{1 \leq r \leq n} S_i^{d_i}$ . Then we find a copy of  $\mathfrak{Grass}_{\mathbf{d}}^T$  as a (closed) subvariety of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ : Indeed, the distinguished projective cover  $P$  of  $T$  can be identified with  $\bigoplus_{r \in \Delta} \Lambda z_r$  for a suitable subset  $\Delta \subseteq \{1, \dots, d\}$ . Then  $\mathfrak{Grass}_{\mathbf{d}}^T$  is evidently isomorphic to the subvariety of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  consisting of the points  $C$  which contain the elements  $z_r$  for  $r \notin \Delta$  such that, moreover,  $C \cap \bigoplus_{r \in \Delta} \Lambda z_r \subseteq \bigoplus_{r \in \Delta} Jz_r$ . In fact, if  $T$  contains simple summands with multiplicity  $\geq 2$ , the encompassing variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  contains infinitely many copies of  $\mathfrak{Grass}_{\mathbf{d}}^T$ . This makes it far more unwieldy than the parametrizing varieties  $\mathfrak{Grass}_{\mathbf{d}}^T$  and  $\mathfrak{Grass}(\mathbb{S})$ , and we will retreat to the latter whenever possible.

Following the model of  $\mathbf{Mod}_{\mathbf{d}}^T$  and  $\mathfrak{Grass}_{\mathbf{d}}^T$ , we further subdivide the varieties  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  in terms of tops and radical layerings.

**Definition 2.7 and Comments.** Let  $T$  and  $\mathbf{P}$  be as above, and let  $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$  be a semisimple sequence with dimension vector  $\mathbf{d}$ . (Remember that by choice of  $\mathbf{P}$ , the top  $\mathbb{S}_0$  of  $\mathbb{S}$  must be  $\leq \mathbf{P}/J\mathbf{P}$ .) We set

$$\begin{aligned} \text{GRASS}_{\mathbf{d}}^T &:= \{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid \mathbf{P}/C \text{ has top } T\} \\ \text{GRASS}(\mathbb{S}) &:= \{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid \mathbb{S}(\mathbf{P}/C) = \mathbb{S}\}. \end{aligned}$$

By Observation 2.9 below,  $\text{GRASS}_{\mathbf{d}}^T$  is locally closed in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  and, by Observation 2.11, so is  $\text{GRASS}(\mathbb{S})$ .

The variety  $\text{GRASS}_{\mathbf{d}}^T$  is the disjoint union of those  $\text{GRASS}(\mathbb{S})$  for which  $\mathbb{S}_0 = T$ , and it contains the copies of  $\mathfrak{Grass}_{\mathbf{d}}^T$  in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  mentioned above. Similarly, the corresponding copies of  $\mathfrak{Grass}(\mathbb{S})$  can all be found in  $\text{GRASS}(\mathbb{S})$ .

We next compare the varieties  $\mathfrak{Grass}_{\mathbf{d}}^T$  and  $\mathfrak{Grass}(\mathbb{S})$  to  $\text{GRASS}_{\mathbf{d}}^T$  and  $\text{GRASS}(\mathbb{S})$  in a couple of small examples.

**Examples 2.8.** (1) Let  $\Lambda = K\Gamma/I$ , where  $\Gamma$  is the quiver  $1 \rightleftarrows 2$  and  $I$  is generated by the paths of length 2. For  $\mathbf{d} = (1, 1)$ , the large variety  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  has two irreducible components, each of which is isomorphic to a copy of  $\mathbb{P}^1$ . One of these components consists the  $\text{Aut}_{\Lambda}(\mathbf{P})$ -orbits of the modules  $\Lambda e_1$  and  $S_1 \oplus S_2$ , the other component parametrizes the modules  $\Lambda e_2$  and  $S_1 \oplus S_2$ . In either case, the projective module  $\Lambda e_i$  has an  $\text{Aut}_{\Lambda}(\mathbf{P})$ -orbit isomorphic to  $\mathbb{A}^1$ , and the orbit of  $S_1 \oplus S_2$  is a singleton. Moreover, each component equals the closure of  $\text{GRASS}_{\mathbf{d}}^{S_i}$  in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

On the other hand,  $\mathfrak{Grass}_{\mathbf{d}}^{S_i}$  consists of the single point  $Je_i$ .

(2) Let  $\Lambda = KQ$ , where  $Q$  is the generalized Kronecker quiver

$$1 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} 2$$

and choose  $\mathbf{d} = (2, 3)$ ,  $T = S_1^2$ , and  $\mathbb{S} = (S_1^2, S_2^3)$ . The distinguished projective cover  $\mathbf{P}$  of  $S_1^2 \oplus S_2^3$  is  $\mathbf{P} = \bigoplus_{1 \leq r \leq 5} \Lambda z_r$ , where  $\Lambda z_r \cong \Lambda e_1$  for  $r = 1, 2$ , and  $\Lambda z_r \cong \Lambda e_2 = S_2$  for

$r = 3, 4, 5$ . The distinguished projective cover of  $T$  is the submodule  $P = \Lambda z_1 \oplus \Lambda z_2$  of  $\mathbf{P}$ . In particular,  $J\mathbf{P}$  and  $JP$  are direct sums of copies of  $S_2$ , and consequently every subspace is a  $\Lambda$ -submodule. In the small Grassmannian setting one obtains

$$\mathfrak{Grass}_{\mathbf{d}}^T = \mathfrak{Grass}(\mathbb{S}) \cong \mathfrak{Gr}(3, JP) \cong \mathfrak{Gr}(3, K^6),$$

whence  $\dim \mathfrak{Grass}_{\mathbf{d}}^T = 9$ . For the large setting, one computes:

$$\text{GRASS}_{\mathbf{d}}(\Lambda) = \mathfrak{Gr}(6, e_2\mathbf{P}) \cong \mathfrak{Gr}(6, K^9)$$

has dimension 18, and  $\text{GRASS}_{\mathbf{d}}^T = \text{GRASS}(\mathbb{S})$  is a dense open subset. Indeed,  $\text{GRASS}_{\mathbf{d}}^T = \{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid \dim(C \cap J\mathbf{P}) = 3\}$ . Openness now follows from the fact that, for any point  $C \in \text{GRASS}_{\mathbf{d}}(\Lambda)$ , the intersection  $C \cap J\mathbf{P}$  has dimension at least 3, and therefore  $\text{GRASS}_{\mathbf{d}}^T$  coincides with the open subvariety  $\{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid \dim(C \cap J\mathbf{P}) < 4\}$ . In particular,  $\text{GRASS}_{\mathbf{d}}^T$  has in turn dimension 18.  $\square$

## 2.E. Useful closed subsets of $\text{GRASS}_{\mathbf{d}}(\Lambda)$ and $\mathfrak{Grass}_{\mathbf{d}}^T$ .

The following fact is straightforward from upper semicontinuity of the maps  $C \mapsto \dim \text{Hom}_{\Lambda}(\mathbf{P}/C, S_i)$  for  $1 \leq i \leq n$ .

**Observation 2.9.** *Given any semisimple module  $T \leq \mathbf{P}/J\mathbf{P}$ , the subset*

$$\{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid T \leq \text{top}(\mathbf{P}/C)\}$$

*of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  is closed. Consequently, the set  $\text{GRASS}_{\mathbf{d}}^T$  is a locally closed subvariety of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .  $\square$*

Next, we introduce a partial order on the (finite) set of semisimple sequences of dimension vector  $\mathbf{d}$ . In analogy with Observation 2.9, it will provide us with a useful array of closed subsets of  $\mathfrak{Grass}_{\mathbf{d}}^T$ .

**Definition 2.10.** Let  $\mathbb{S}$  and  $\mathbb{S}'$  be two semisimple sequences with the same dimension vector. We say that  $\mathbb{S}'$  *dominates*  $\mathbb{S}$  and write  $\mathbb{S} \leq \mathbb{S}'$  if and only if  $\bigoplus_{l \leq r} \mathbb{S}_l \leq \bigoplus_{l \leq r} \mathbb{S}'_l$  for all  $r \geq 0$ .

Roughly speaking,  $\mathbb{S}'$  dominates  $\mathbb{S}$  if and only if  $\mathbb{S}'$  results from  $\mathbb{S}$  through a finite sequence of leftward shifts of simple summands of  $\bigoplus_{0 \leq l \leq L} \mathbb{S}_l$  in the layering provided by the  $\mathbb{S}_l$ .

In intuitive terms, the next observation says that the simple summands in the radical layers of the modules represented by  $\text{GRASS}(\mathbb{S})$  are only “upwardly mobile” as one passes to modules in the boundary of the closure of  $\text{GRASS}(\mathbb{S})$  in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

**Observation 2.11.** *Suppose that  $\mathbb{S}$  is a semisimple sequence with dimension vector  $\mathbf{d}$ . Then the union  $\bigcup_{\mathbb{S}' \geq \mathbb{S}, \dim \mathbb{S}' = \mathbf{d}} \text{GRASS}(\mathbb{S}')$  is closed in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ . Analogously, the union  $\bigcup_{\mathbb{S}' \geq \mathbb{S}, \dim \mathbb{S}' = \mathbf{d}} \mathbf{Mod}(\mathbb{S}')$  is closed in  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ .*

*In particular,  $\text{GRASS}(\mathbb{S})$  is a locally closed subvariety of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .*

*Proof.* For simplicity, we posit that all semisimple sequences mentioned in the proof have dimension vector  $\mathbf{d}$ .

By Proposition 2.5, it suffices to prove the claim in the Grassmannian setting. We write  $J^l \mathbf{P} / J^{l+1} \mathbf{P} = \bigoplus_{1 \leq i \leq n} S_i^{t_{li}}$  and  $\mathbb{S}_l = \bigoplus_{1 \leq i \leq n} S_i^{s_{li}}$  for each  $l \in \{0, \dots, L\}$ ; note that  $s_{li} \leq t_{li}$ , since  $\mathbb{S}_l \leq J^l \mathbf{P} / J^{l+1} \mathbf{P}$  for all  $l \geq 0$  by the definition of a semisimple sequence with a top contained in  $\mathbf{P} / J\mathbf{P}$ . Moreover, for each vertex  $e_i$  of  $\Gamma$ , we consider the following partial flag of subspaces of the  $K$ -space  $\mathbf{P}$ :

$$e_i J^L \mathbf{P} \subseteq e_i J^{L-1} \mathbf{P} \subseteq \dots \subseteq e_i J \mathbf{P} \subseteq e_i \mathbf{P}.$$

Then the union of the  $\text{GRASS} \mathbb{S}'$ , where  $\mathbb{S}'$  traces the semisimple sequences dominating  $\mathbb{S}$ , coincides with  $\text{GRASS}_{\mathbf{d}}(\Lambda) \cap \left( \bigcap_{1 \leq i \leq n} V_i \right)$ , where

$$V_i = \{C \in \mathfrak{Gr}(\dim \mathbf{P} - d, \mathbf{P}) \mid \dim(C \cap e_i J^l \mathbf{P}) \geq \sum_{k \geq l} (t_{ki} - s_{ki}) \text{ for all } 1 \leq l \leq L\}.$$

The latter sets are well known to be closed in  $\mathfrak{Gr}(\dim \mathbf{P} - d, \mathbf{P})$ , and our first claim follows.

To derive that  $\text{GRASS}(\mathbb{S})$  is locally closed in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ , let  $\mathbb{S}^{(1)}, \dots, \mathbb{S}^{(u)}$  be the semisimple sequences which are strictly larger than  $\mathbb{S}$ . Then

$$\text{GRASS}(\mathbb{S}) = \left( \bigcup_{\mathbb{S}' \geq \mathbb{S}} \text{GRASS}(\mathbb{S}') \right) \setminus \left( \bigcup_{1 \leq i \leq u} \bigcup_{\mathbb{S}' \geq \mathbb{S}^{(i)}} \text{GRASS}(\mathbb{S}') \right).$$

Since there are only finitely many semisimple sequences of any given dimension vector, this proves our final claim.  $\square$

**Corollary 2.12.** (cf. [9], Observation 3.4.) *Suppose that  $\mathbb{S}$  is a semisimple sequence with dimension vector  $\mathbf{d}$  and top  $T$ . Then the union*

$$\bigcup_{\mathbb{S}' \geq \mathbb{S}, \mathbb{S}'_0 = T, \underline{\dim} \mathbb{S}' = \mathbf{d}} \mathfrak{Grass}(\mathbb{S}')$$

*is closed in  $\mathfrak{Grass}_d^T$ . (As before,  $\mathbb{S}'_0$  is the semisimple module in the 0-th slot of the sequence  $\mathbb{S}'$ .)*  $\square$

## 2.F. Structure of the acting automorphism groups.

The presence of a, usually large, unipotent radical in the acting automorphism groups  $\text{Aut}_{\Lambda}(Q)$  (e.g., for  $Q = P$  or  $Q = \mathbf{P}$ ) will turn out to be of advantage in making a different arsenal of methods applicable than are available for the action of the reductive group  $\text{GL}_d$  in the classical affine setting. The following facts are straightforward, but very useful.

**Proposition 2.13 and terminology.** *Let  $Q$  be any finitely generated projective  $\Lambda$ -module. Then the automorphism group  $\text{Aut}_{\Lambda}(Q)$  is isomorphic to the semidirect product*

$$(\text{Aut}_{\Lambda}(Q))_u \rtimes \text{Aut}_{\Lambda}(Q/JQ),$$

where  $(\mathrm{Aut}_\Lambda(Q))_u$  is the unipotent radical of  $\mathrm{Aut}_\Lambda(Q)$ .

Moreover,  $(\mathrm{Aut}_\Lambda(Q))_u$  equals  $\{\mathrm{id}_Q + g \mid g \in \mathrm{Hom}_\Lambda(Q, JQ)\}$ .

If  $Q = \bigoplus_{1 \leq r \leq m} \Lambda z_r$  with distinguished top elements  $z_r$ , then the following subgroup  $\mathcal{T}_0$  of  $\mathrm{Aut}_\Lambda(Q)$  (and of  $\mathrm{Aut}_\Lambda(Q/JQ)$ , up to isomorphism) will be referred to as the distinguished torus in  $\mathrm{Aut}_\Lambda(Q)$ : Namely,

$$\mathcal{T}_0 = \{f \in \mathrm{Aut}_\Lambda(Q) \mid z_1, \dots, z_m \text{ are eigenvectors of } f\}. \quad \square$$

As a consequence, we can split up the investigation of orbits and orbit closures in the above setting into two tasks: exploring the action of the unipotent group  $(\mathrm{Aut}_\Lambda(Q))_u$  and that of the reductive group  $\mathrm{Aut}_\Lambda(Q/JQ) \cong \prod_{1 \leq i \leq n} \mathrm{GL}_{m_i}(K)$ , where  $m_i$  is the multiplicity of the simple module  $S_i$  in the top of  $Q$ .

In particular, the following results will be useful on many occasions. The first is due to Rosenlicht [12], the second was first proved by Kostant for  $K = \mathbb{C}$  and subsequently generalized to arbitrary base fields by Rosenlicht [11], Theorem 2; cf. [5], Proposition 4.10.

**Theorem 2.14.** *Let  $G$  be a connected unipotent algebraic group and  $V$  a variety on which  $G$  acts morphically. Then:*

- (1) *All  $G$ -orbits of  $V$  are isomorphic to affine spaces.*
- (2) *If  $V$  is quasi-affine, then all  $G$ -orbits are closed in  $V$ .  $\square$*

To apply part (1), we typically take  $V$  to be the variety  $[\mathrm{GRASS}_d(\Lambda)]_Q$ , where  $Q$  is a projective module with distinguished sequence  $z_1, \dots, z_m$  of top elements as in Definition 2.3 and Proposition 2.13, and let  $G = (\mathrm{Aut}_\Lambda(Q))_u$  be the unipotent radical of  $\mathrm{Aut}_\Lambda(Q)$ . For  $C \in [\mathrm{GRASS}_d(\Lambda)]_Q$ , we spell out an explicit isomorphism of varieties  $(\mathrm{Aut}_\Lambda(Q))_u.C \rightarrow \mathbb{A}^{\mathfrak{m}}$ , where  $\mathfrak{m}$  is the dimension of the considered orbit. Setting  $M = Q/C$ , we obtain in analogy to [9], Observation 3.2:

$$\mathfrak{m} = \dim \mathrm{Hom}_\Lambda(Q, JM) - \dim \mathrm{Hom}_\Lambda(M, JM).$$

The isomorphism we describe will be used in the proof of Proposition 4.4. Note that the stabilizer of  $C$  in  $(\mathrm{Aut}_\Lambda(Q))_u$  equals the set of all automorphisms of the form  $\mathrm{id}_Q + g$ , where  $g$  runs through the subspace

$$\mathrm{Stab}_{\mathrm{Hom}_\Lambda(Q, JQ)}(C) = \{g \in \mathrm{Hom}_\Lambda(Q, JQ) \mid g(C) \subseteq C\}.$$

Pick homomorphisms  $g_1, \dots, g_{\mathfrak{m}} \in \mathrm{Hom}_\Lambda(Q, JQ)$  which constitute a  $K$ -basis for  $\mathrm{Hom}_\Lambda(Q, JQ)$  modulo  $\mathrm{Stab}_{\mathrm{Hom}_\Lambda(Q, JQ)}(C)$ . Then

$$\mathbb{A}^{\mathfrak{m}} \rightarrow (\mathrm{Aut}_\Lambda(Q))_u.C, \quad (a_i) \mapsto (\mathrm{id}_Q + \sum_i a_i g_i).C$$

is an isomorphism as desired. To describe its image in concrete computations, it is often advantageous to content oneself with a generating set  $(g_i)$  for  $\mathrm{Hom}_\Lambda(Q, JQ)$  modulo  $\mathrm{Hom}_\Lambda(Q, C \cap JQ)$ . Particularly convenient are the maps  $g_{r,i}$  with  $g_{r,i}(z_s) = p_{r,i} z_{j(r,i)}$  for  $s = r$  and  $g_{r,i}(z_s) = 0$  for  $s \neq r$ , where  $p_{r,i} \widehat{z}_{j(r,i)}$  trace those paths in a skeleton of  $M = Q/C$  which have positive length and end in  $e(r)$ .

### 3. SKELETA OF MODULES AND AFFINE CHARTS FOR $\text{GRASS}_{\mathbf{d}}(\Lambda)$ AND $\mathfrak{Grass}_{\mathbf{d}}^T$

The affine subvarieties  $\text{GRASS}(\sigma)$  covering  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  which we will next introduce and scrutinize are generalizations of those that were previously defined in [1] for  $\mathfrak{Grass}_{\mathbf{d}}^T$ ; predecessors for special choices of  $T$  were considered in [3, 8, 9]. These affine charts are distinguished by the following three assets: First, they are always stable under the action of the unipotent radical  $(\text{Aut}_{\Lambda}(\mathbf{P}))_u$  of the acting automorphism group. This brings the theory of unipotent group actions to bear on the individual patches; see, e.g., Theorem 2.14. In fact, the  $\text{GRASS}(\sigma)$  are even stable under the action of  $(\text{Aut}_{\Lambda}(\mathbf{P}))_u \rtimes \mathcal{T}_0$ , where  $\mathcal{T}_0$  is the distinguished maximal torus in  $\text{Aut}_{\Lambda}(\mathbf{P})$  introduced in Proposition 2.13. Another plus lies in the tight connection between these affine charts and the structural features of the modules they represent. The  $\text{GRASS}(\sigma)$  consist precisely of the points in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  which correspond to modules sharing a “special path basis”  $\sigma$ , which we will call a *skeleton*; such a skeleton pins down other structural data, such as the radical layering, of the pertinent module. Finally, for large classes of algebras beyond the hereditary ones, the  $\text{GRASS}(\sigma)$  are rational.

The affine cover  $(\text{GRASS}(\sigma))_{\sigma}$  of  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  will turn out to consist of intersections of Schubert cells with the varieties  $\text{GRASS}(\mathbb{S})$ . In particular, we hence find it to be the union of affine covers of the individual subvarieties  $\text{GRASS}(\mathbb{S})$ . (On the side, we mention that, while the classical affine Schubert cells of  $\mathfrak{Gr}(\dim \mathbf{P} - d, \mathbf{P})$  obviously intersect  $\text{GRASS}_{\mathbf{d}}(\Lambda)$  in affine sets – the intersection with the Schubert cell that accompanies any skeleton  $\sigma$  is labeled  $\text{SCHU}(\sigma)$  below – these intersections are hardly ever  $(\text{Aut}_{\Lambda}(\mathbf{P}))_u$ -stable; nor are they stable under other relevant subgroups of  $\text{Aut}_{\Lambda}(\mathbf{P})$  in general.) We point out that skeleta of modules are generic attributes in the following sense: Every irreducible component of the variety  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  contains a dense open subset such that all modules in this subset have the same skeleton ([1]; combine Observations 2.1 and 2.4 with Theorem 3.8). Finally, there is a fast algorithm [2] which provides the polynomials defining the distinguished affine patches, starting from a presentation of  $\Lambda$  in terms of quiver and relations. The package is coupled with an algorithm that decomposes the varieties  $\mathfrak{Grass}(\sigma)$  into irreducible components. It is on the level of the  $\text{GRASS}(\sigma)$  covering  $\text{GRASS}(\mathbb{S})$  (and the  $\mathfrak{Grass}(\sigma)$  covering  $\mathfrak{Grass}(\mathbb{S})$ ) that links between algebraic features of the modules being parametrized on one hand, and the geometry of the parametrizing variety on the other, become most accessible.

A downside of the  $\text{GRASS}(\sigma)$ : In general, these affine patches are not open in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ , but only open in the subvarieties  $\text{GRASS}(\mathbb{S})$ . Moreover, the corresponding  $\text{Aut}_{\Lambda}(\mathbf{P})$ -orbits have comparatively large dimension and are far less manageable than the  $\text{Aut}_{\Lambda}(P)$ -orbits of the variety  $\mathfrak{Grass}_{\mathbf{d}}^T$ , where  $T$  equals the top  $\mathbb{S}_0$  of  $\mathbb{S}$ . One counters it by moving back and forth between the large and small settings.

#### 3.A. Definition of skeleta and a first example.

Roughly speaking, skeleta allow us to carry some of the benefits of the path-length grading of the projective  $KT$ -modules to arbitrary  $\Lambda$ -modules.

**Setting:** As in Section 2.B, we fix a semisimple module  $T \in \Lambda\text{-mod}$  with dimension vector  $\mathbf{t}$  and total dimension  $t$ . Moreover,  $\mathbf{d}$  continues to denote a dimension vector  $\geq \mathbf{t}$  of total dimension  $d$ . As in Section 2.D, we fix a projective  $\Lambda$ -module  $Q$  such that

$T \leq Q/JQ \leq \bigoplus_{1 \leq i \leq n} S_i^{d_i}$ , and pair it with a distinguished sequence of top elements  $z_1, \dots, z_m$  for  $Q$  such that  $z_r = e(r)z_r$  for primitive idempotents  $e(r) \in \{e_1, \dots, e_n\}$ . In the special cases where  $Q$  is the distinguished projective cover of  $T$ , or of the semisimple module  $\bigoplus_{1 \leq i \leq n} S_i^{d_i}$  of dimension vector  $\mathbf{d}$ , we denote  $Q$  by  $P$  or  $\mathbf{P}$ , respectively.

Now write  $\widehat{Q} = \bigoplus_{1 \leq r \leq m} K\Gamma \widehat{z}_r$  for the projective  $K\Gamma$ -module covering the projective  $\Lambda$ -module  $Q = \bigoplus_{1 \leq r \leq m} \Lambda z_r$ ; here each  $\widehat{z}_r$  is normed by the same vertex of the quiver  $\Gamma$  as  $z_r$ . In other words,  $\widehat{Q}/I\widehat{Q} \cong Q$  under the canonical  $K\Gamma$ -epimorphism  $\widehat{Q} \rightarrow Q$  which sends  $\widehat{z}_r$  to  $z_r$ . In the two most frequently considered instances, we write  $\widehat{P} = \bigoplus_{1 \leq r \leq t} K\Gamma \widehat{z}_r$  and  $\widehat{\mathbf{P}} = \bigoplus_{1 \leq r \leq d} K\Gamma \widehat{z}_r$ , respectively.

By a *path in  $\widehat{Q}$  starting in  $\widehat{z}_r$*  we mean any element  $p' = p\widehat{z}_r \in \widehat{Q}$ , where  $p$  is a path in  $K\Gamma$  starting in the vertex  $e(r)$ . The *length* of  $p'$  is defined to be that of  $p$ ; ditto for the *endpoint* of  $p'$ . If  $p_1$  is an initial subpath of  $p$ , meaning that  $p = p_2 p_1$  for paths  $p_1, p_2$ , we call  $p_1 \widehat{z}_r$  an *initial subpath of  $p\widehat{z}_r$* . So, in particular,  $\widehat{z}_r = e(r)\widehat{z}_r$  is an initial subpath of length 0 of any path  $p\widehat{z}_r$  in  $\widehat{Q}$ .

The reason why we do not identify  $p' = p\widehat{z}_r$  with  $p$  lies in the fact that we want to distinguish between  $p\widehat{z}_r$  and  $p\widehat{z}_s$  for  $r \neq s$  but  $e(r) = e(s)$ . A priori, we moreover distinguish between a path  $p\widehat{z}_r \in \widehat{Q}$  and the corresponding residue class  $pz_r \in Q = \widehat{Q}/I\widehat{Q}$  in order to keep an unambiguous notion of length. However, in the sequel, we will often not uphold the distinction between  $p\widehat{z}_r$  and  $pz_r$ , unless there is a need to emphasize well-definedness of path lengths.

We always have distinguished embeddings  $P \subseteq Q \subseteq \mathbf{P}$  (those respecting the distinguished top elements carrying the same label), and correspondingly,  $\widehat{P} \subseteq \widehat{Q} \subseteq \widehat{\mathbf{P}}$ . Some of the theory we explicitly develop only for the special cases  $Q = P$  and  $Q = \mathbf{P}$ , in order to pare down the setting to a more tightly determined one for clarity. Unless we emphasize the contrary, the results carry over to the general situation with analogous proofs.

### Definition 3.1 and Conventions.

(1) An (*abstract*) *skeleton with top  $T$*  in  $\widehat{Q}$  is a set  $\sigma$  of paths of lengths at most  $L$  in  $\widehat{Q}$ , which contains  $t$  paths of length zero, say  $\widehat{z}_{i_1}, \dots, \widehat{z}_{i_t}$  such that  $\bigoplus_{1 \leq r \leq t} \Lambda z_{i_r}$  is a  $\Lambda$ -projective cover of  $T$ ; moreover,  $\sigma$  is required to be closed under initial subpaths in the following sense: whenever  $p_2 p_1 \widehat{z}_r \in \sigma$ , then  $p_1 \widehat{z}_r \in \sigma$ . (We usually suppress reference to the  $K\Gamma$ -projective module  $\widehat{Q}$ , when our choice of the  $\Lambda$ -projective module  $Q$  is clear.)

We say that  $\sigma$  has *dimension vector  $\mathbf{d}$*  if, for each  $i$  between 1 and  $n$ , the set  $\sigma$  contains exactly  $d_i$  distinct paths ending in the vertex  $e_i$ .

Alternatively, we view a skeleton  $\sigma$  as a forest of  $t$  tree graphs, where each tree consists of the paths in  $\sigma$  starting in a fixed top element  $\widehat{z}_r$ ; see Example 3.4 below and the remarks preceding it.

Note: In case  $Q = P$  is the distinguished projective cover of  $T$ , any abstract skeleton with top  $T$  in  $\widehat{P}$  contains the full collection of paths of length zero in  $\widehat{P}$ . In light of the canonical embeddings  $\widehat{P} \subseteq \widehat{Q} \subseteq \widehat{\mathbf{P}}$ , we routinely view a skeleton in  $\widehat{P}$  as a subset of  $\widehat{\mathbf{P}}$ , whenever convenient.



Further conventions:

- By  $\sigma_l$  we denote the set of paths of length  $l$  in  $\sigma$ .
- When we pass from the  $K\Gamma$ -module  $\widehat{Q}$  to the  $\Lambda$ -module  $Q$  by factoring out  $I\widehat{Q}$ , we identify  $\widehat{z}_{i_r}$  with  $z_{i_r}$  and view  $\sigma$  as the subset

$$\{pz_{i_r} \in Q \mid p\widehat{z}_{i_r} \in \sigma, r \leq t\}$$

of  $Q$

(2) Let  $\sigma$  be an abstract skeleton with top  $T$  in  $\widehat{P}$  (or, more generally, in  $\widehat{Q}$ ), and  $M$  a  $\Lambda$ -module.

We call  $\sigma$  a *skeleton of  $M$*  if there exists a full sequence of top elements  $x_1, \dots, x_t$  of  $M$ , together with a  $K\Gamma$ -epimorphism  $f : \widehat{P} \rightarrow M$  (resp.,  $\widehat{Q} \rightarrow M$ ) satisfying  $f(\widehat{z}_{i_r}) = x_r$  for all  $r$  such that, for each  $l \in \{0, \dots, L\}$ , the set

$$f(\sigma_l) = \{f(p\widehat{z}_{i_r}) \mid p\widehat{z}_{i_r} \in \sigma_l\} = \{(p + I)x_r \mid r \leq t, p\widehat{z}_{i_r} \in \sigma_l\}$$

induces a  $K$ -basis for the radical layer  $J^l M / J^{l+1} M$ . In this situation, we also say that  $\sigma$  is a *skeleton of  $M$  relative to the sequence  $x_1, \dots, x_t$  of top elements*, and observe that the union over  $l \leq L$  of the above sets  $f(\sigma_l)$ , namely  $\{(q + I)x_r \mid r \leq t, q\widehat{z}_{i_r} \in \sigma\}$ , is a  $K$ -basis for  $M$ . Thus the skeleta of  $M$  are  $K$ -bases which are closely tied to the quiver presentation of  $\Lambda$ .

If  $M$  has skeleton  $\sigma \subseteq \widehat{Q}$ , then clearly  $M$  is an epimorphic image of  $Q$ . If  $M = Q/C$ , we call  $\sigma$  a *distinguished skeleton of  $M$*  provided that  $\sigma$  is a skeleton of  $M$  relative to the distinguished sequence  $z_{i_1} + C, \dots, z_{i_t} + C$  of top elements of  $Q/C$ .

(3) If  $Q = P$  is the projective cover of  $T$ , we write

$$\mathfrak{Grass}(\sigma) = \{C \in \mathfrak{Grass}_d^T \mid \sigma \text{ is a distinguished skeleton of } P/C\},$$

as in [1].

If, on the other hand,  $Q = \mathbf{P}$  is the projective cover of  $\bigoplus_{1 \leq i \leq n} S_i^{d_i}$ , we define

$$\text{GRASS}(\sigma) = \{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid \sigma \text{ is a distinguished skeleton of } \mathbf{P}/C\}.$$

As we pointed out before,  $\mathfrak{Grass}(\sigma)$  and  $\text{GRASS}(\sigma)$  parametrize the same collection of isomorphism types of modules. The set of all skeleta of a  $d$ -dimensional module  $M$  is clearly an isomorphism invariant of  $M$ . This set is always nonempty and finite, the latter being due to our requirement that the distinguished projective modules  $Q$  have tops contained in the semisimple module  $(\Lambda/J)^d$ . Moreover, if  $M = Q/C$ , this set contains at least one distinguished candidate. We record a slightly upgraded variant of the existence statement, leaving the easy proof to the reader.

**Observation 3.2.** *Suppose  $M \in \Lambda\text{-mod}$  has top  $T$ , and let  $x_1, \dots, x_t$  be a full sequence of top elements of  $M$ . Then  $M$  has at least one skeleton in  $\widehat{P}$  relative to  $x_1, \dots, x_t$ . Moreover, whenever a module  $\tilde{M}$  has the same top as  $M$  and maps epimorphically onto  $M$ , any skeleton of  $M$  can be extended to a skeleton of  $\tilde{M}$ . In particular, every skeleton of  $M$  is contained in a skeleton of  $P$ .*

*Analogously, any module  $M$  with dimension vector  $\mathbf{d}$  has at least one skeleton in  $\widehat{\mathbf{P}}$ , and every such skeleton is contained in a skeleton of  $\mathbf{P}$  in  $\widehat{\mathbf{P}}$ .  $\square$*

Our definition of a skeleton coincides in essence with that given in [8] and [9] for the situation  $Q = P$  and a squarefree top  $T = P/JP$ . However, in that special case, it is unnecessary to hook up the elements of an abstract skeleton  $\sigma$  with top elements of the  $KT$ -module  $\widehat{P}$ , since the dependence on specific sequences of top elements disappears. In particular, every skeleton of  $P/C$  is distinguished in the case of squarefree  $T$ : Indeed, if  $y_1, \dots, y_t$  is any sequence of top elements of  $P$  such that each  $y_i$  is normed by the same primitive idempotent as  $z_i$ , the  $y_i$  can differ from the  $z_i$  only by a nonzero constant factor and a summand in  $JP$ ; consequently, any skeleton of  $P/C$  relative to  $y_1, \dots, y_t$  is also a skeleton relative to the distinguished sequence  $z_1, \dots, z_t$ .

Again, let  $Q$  be either  $P$  or  $\mathbf{P}$ , and let  $\sigma$  be an abstract skeleton with dimension vector  $\mathbf{d}$  and an arbitrary top  $T$ . Suppose that  $M = Q/C$ . Then  $\sigma$  is a (not necessarily distinguished) skeleton of  $M$  if and only if  $M \cong Q/D$  for *some* point  $D$  in  $\mathfrak{Grass}(\sigma)$  (resp., in  $\text{GRASS}(\sigma)$ ). However,  $\text{Aut}_\Lambda(P).C \cap \mathfrak{Grass}(\sigma) \neq \emptyset$  need not imply  $C \in \mathfrak{Grass}(\sigma)$ , and an analogous caveat pertains to the big setting. On the other hand, the  $\mathfrak{Grass}(\sigma)$  (resp.,  $\text{GRASS}(\sigma)$ ) do enjoy the following partial stability under the  $\text{Aut}_\Lambda(P)$ -action (resp.,  $\text{Aut}_\Lambda(\mathbf{P})$ -action):

**Observation 3.3 and Convention.** The subvarieties  $\mathfrak{Grass}(\sigma)$  of  $\mathfrak{Grass}_\mathbf{d}^T$  are stable under the action of  $G = (\text{Aut}_\Lambda(P))_u \rtimes \mathcal{T}_0$ , where  $(\text{Aut}_\Lambda(P))_u$  is the unipotent radical of  $\text{Aut}_\Lambda(P)$  and  $\mathcal{T}_0$  the distinguished torus in  $\text{Aut}_\Lambda(P)$  (see Proposition 2.13); we write the elements of  $\mathcal{T}_0$  in the form  $(a_1, \dots, a_t) \in (K^*)^t$ , the latter standing for the automorphism that sends  $z_r$  to  $a_r z_r$ .

Analogously, the sets  $\text{GRASS}(\sigma)$  are stable under the action of the group  $G = (\text{Aut}_\Lambda(\mathbf{P}))_u \rtimes \mathcal{T}_0$ , where  $(\text{Aut}_\Lambda(\mathbf{P}))_u$  and  $\mathcal{T}_0$  are the unipotent radical and distinguished torus in  $\text{Aut}_\Lambda(\mathbf{P})$ .

*Proof.* We only address the small scenario  $\mathfrak{Grass}_\mathbf{d}^T$ , the arguments for the big being analogous. Suppose  $C \in \mathfrak{Grass}(\sigma)$ , meaning that  $\sigma$  is a distinguished skeleton of  $P/C$ . Then  $\sigma$  remains a skeleton of  $P/C$  after passage from our given distinguished sequence  $z_1, \dots, z_t$  of top elements of  $P$  to a new sequence of the form  $g.z_1, \dots, g.z_t$  for any  $g \in G$ . But this is tantamount to saying that  $\sigma$  is a distinguished skeleton of  $P/(g^{-1}.C)$  relative to the original sequence. In other words,  $g^{-1}.C \in \mathfrak{Grass}(\sigma)$ .  $\square$

Any abstract skeleton  $\sigma$  can be communicated by means of an undirected graph which is a *forest*, that is, a finite disjoint union of tree graphs: There are  $t$  trees if  $\sigma$  is a skeleton with top  $T$ , one for each  $r \leq t$ ; here  $\widehat{z}_r$ , identified with  $e(r)$ , represents the root of the  $r$ -th tree, recorded in the top row of the graph. The paths  $p\widehat{z}_r$  of positive length in  $\sigma$  are

represented by edge paths of positive length. Instead of formalizing this convention, we will illustrate it in Example 3.4(2).

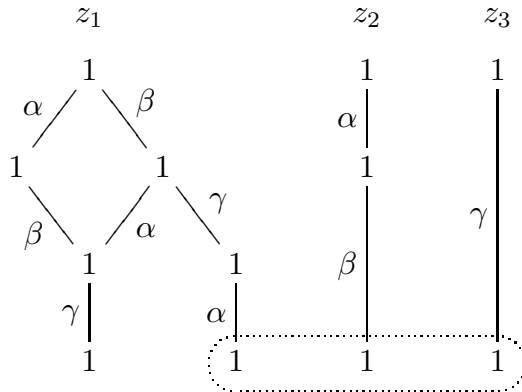
In general, neither  $\mathbf{Grass}_d^T$  nor  $\mathbf{Grass}_d(\Lambda)$  has an affine cover which is closed under the full  $\text{Aut}_\Lambda(P)$ -action (resp.,  $\text{Aut}_\Lambda(\mathbf{P})$ -action), as witnessed by the first of the following examples. Indeed, the existence of such a cover would force all orbits to be quasi-affine.

### Examples 3.4.

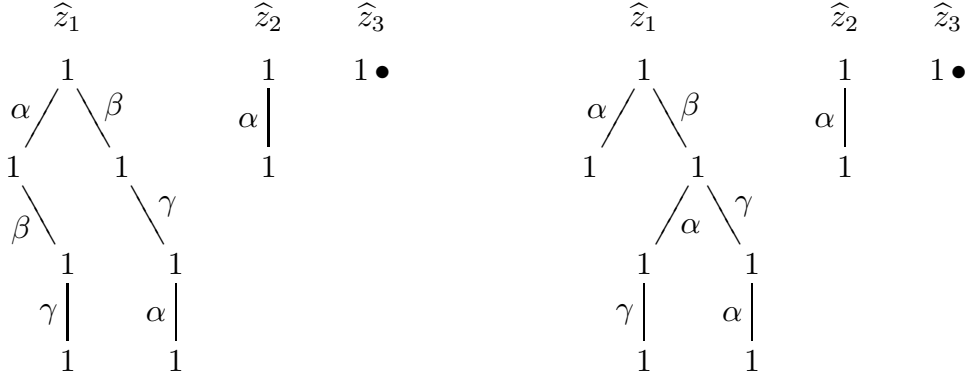
(1) Let  $\Gamma$  be the quiver  $1 \rightarrow 2$ ,  $\mathbf{d} = (2, 1)$ ,  $T = S_1^2$  and  $\mathbb{S} = (S_1^2, S_2)$ . Then  $\mathbf{Grass}_d^T \cong \mathbb{P}^1$  consists of a single  $\text{Aut}_\Lambda(P)$ -orbit, where  $P = \Lambda z_1 \oplus \Lambda z_2 \cong (\Lambda e_1)^2$ . Here  $\mathbf{P} = P \oplus \Lambda z_3$ , where  $\Lambda z_3 = \Lambda e_2 = S_2$ , and  $\mathbf{Grass}_d(\Lambda) \cong \mathbb{P}^2$  consists of two  $\text{Aut}_\Lambda(\mathbf{P})$ -orbits. One of them equals  $\mathbf{Grass}(\mathbb{S})$ , the other is a singleton, namely  $\{J\mathbf{P}\}$ . Clearly, the big orbit again fails to be quasi-affine.

(2) In the following, we give an example of a module with several distinct skeleta, two of which we will present in graphical format. Let  $\Lambda = K\Gamma/I$ , where  $\Gamma$  is the quiver with a single vertex, labeled 1, and three loops,  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $I \subseteq K\Gamma$  the ideal generated by  $\alpha^2$ ,  $\beta^2$ ,  $\gamma^2$ , and all paths of length 4.

We denote by  $P = \bigoplus_{1 \leq r \leq 3} \Lambda z_r$  the distinguished projective cover of  $T = S_1^3$ , let  $d = 11$ , and consider the  $d$ -dimensional module  $M = P/C$ , where  $C \in \mathbf{Grass}_d^T$  is generated by  $\gamma z_1$ ,  $\gamma \alpha z_1$ ,  $\beta z_2$ ,  $\gamma z_2$ ,  $\gamma \alpha z_2$ ,  $\alpha z_3$ ,  $\beta z_3$ ,  $(\beta \alpha - \alpha \beta) z_1$ ,  $\alpha \gamma \beta z_1 - \beta \alpha z_2 - \gamma z_3$ , and all elements of the form  $p z_2$ ,  $q z_3$ , where  $p$  is a path of length 3 and  $q$  a path of length 2 in  $\Gamma$ . The module  $M$  can be visualized as follows:



Here we use the graphing technique introduced formally in [1] and informally in [7]. In particular, the grouping of vertices inside the dotted curve means that the images in  $M$  of the paths  $\alpha \gamma \beta z_1$ ,  $\beta \alpha z_2$ ,  $\gamma z_3$  in  $P$  are  $K$ -linearly dependent, whereas any two of them are  $K$ -linearly independent. Up to permutations of the top elements  $\hat{z}_r$  of the projective  $K\Gamma$ -module  $\hat{P} = \bigoplus_{1 \leq r \leq 3} K\Gamma \hat{z}_r$ , the module  $M$  has precisely two skeleta in  $\hat{P}$ . Namely,



Each of the two skeleta consists of all the paths  $p\widehat{z}_r$  that occur as edgepaths in one of the 3 trees on reading them from top to bottom.  $\square$

### 3.B. $\mathfrak{Grass}(\mathbb{S})$ and $\text{GRASS}(\mathbb{S})$ as unions of $\mathfrak{Grass}(\sigma)$ 's and $\text{GRASS}(\sigma)$ 's, respectively.

Suppose that  $\mathbb{S}$  is a semisimple sequence with dimension vector  $\mathbf{d}$  and top  $T$ . We continue to denote by  $P$  and  $\mathbf{P}$  the distinguished projective covers of  $T$  and  $\bigoplus_{1 \leq i \leq n} S_i^{d_i}$ , respectively. We next give a necessary condition for nontriviality of the intersections  $\mathfrak{Grass}(\sigma) \cap \mathfrak{Grass}(\mathbb{S})$  and  $\text{GRASS}(\sigma) \cap \text{GRASS}(\mathbb{S})$ .

**Definition 3.5.** Keep the notation of Definition 3.1. Given a semisimple sequence  $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ , we call a skeleton  $\sigma \subseteq \widehat{Q}$  *compatible with  $\mathbb{S}$*  if, for each  $l \leq L$  and  $i \leq n$ , the number of paths in  $\sigma_l$  ending in the vertex  $e_i$  coincides with the multiplicity of the simple module  $S_i$  in  $\mathbb{S}_l$ .

In case  $Q = P$ , this compatibility concept coincides with the one introduced in [1]. If  $\mathfrak{Grass}(\mathbb{S}) \neq \emptyset$  (or, equivalently,  $\text{GRASS}(\mathbb{S}) \neq \emptyset$ ), there generally are numerous (even though only finitely many) skeleta compatible with  $\mathbb{S}$ . On the other hand, there is only one semisimple sequence  $\mathbb{S}$  with which a given skeleton is compatible.

Clearly, each abstract skeleton in  $\widehat{Q}$  which is compatible with  $\mathbb{S}$  shares top and dimension vector with  $\mathbb{S}$ . Moreover, given a  $\Lambda$ -module  $M$ , each skeleton of  $M$  is compatible with the radical layering  $\mathbb{S}(M)$  of  $M$ ; this is an immediate consequence of the definitions. Consequently, compatibility of  $\sigma$  with  $\mathbb{S}$  is a necessary condition for the intersection  $\mathfrak{Grass}(\sigma) \cap \mathfrak{Grass}(\mathbb{S})$  (resp.,  $\text{GRASS}(\sigma) \cap \text{GRASS}(\mathbb{S})$ ) to be nonempty. More precisely:

**Observation 3.6.** The following statements are equivalent for  $M \in \Lambda\text{-mod}$  and a skeleton  $\sigma$  in  $\widehat{Q}$ :

- $\sigma$  is a skeleton of  $M$ .
- $\sigma$  is compatible with  $\mathbb{S}(M)$ , and there exists a  $KT$ -epimorphism  $f : \widehat{Q} \rightarrow M$  such that  $f(\sigma)$  is a  $K$ -basis for  $M$ .  $\square$

In particular, we glean: If  $\mathfrak{Grass}(\sigma) \neq \emptyset$ , then  $\sigma$  is compatible with  $\mathbb{S}$  precisely when  $\mathfrak{Grass}(\sigma) \subseteq \mathfrak{Grass}(\mathbb{S})$  (equivalently, when  $\text{GRASS}(\sigma) \subseteq \text{GRASS}(\mathbb{S})$ ).

If  $Q = P$ , we denote by  $\text{Schu}(\sigma)$  the intersection of  $\mathfrak{Grass}_{\mathbf{d}}^T$  with the big Schubert cell in the classical Grassmannian  $\mathfrak{Gr}(\dim P - d, JP)$  that corresponds to  $\sigma$ . In other words,

$\text{Schu}(\sigma)$  consists of all points  $C \in \mathfrak{Grass}_{\mathbf{d}}^T$  with the property that

$$JP = C \oplus \bigoplus_{p\hat{z}_r \in \sigma, p\hat{z}_r \text{ of positive length}} Kpz_r.$$

The big counterpart to  $\text{Schu}(\sigma)$  obtained for  $Q = \mathbf{P}$  will be denoted by  $\text{SCHU}(\sigma)$ . It lives in  $\mathfrak{Gr}(\dim \mathbf{P} - d, \mathbf{P})$ . Namely,

$$\text{SCHU}(\sigma) = \{C \in \text{GRASS}_{\mathbf{d}}(\Lambda) \mid \mathbf{P} = C \oplus \bigoplus_{p\hat{z}_r \in \sigma} Kpz_r\}.$$

In particular,  $\text{Schu}(\sigma)$  and  $\text{SCHU}(\sigma)$  are open subsets of  $\mathfrak{Grass}_{\mathbf{d}}^T$  and  $\text{GRASS}_{\mathbf{d}}^T$ , respectively.

**Observation 3.7.** Suppose that  $\sigma$  is a skeleton which is compatible with the semisimple sequence  $\mathbb{S}$ . Then

$$\mathfrak{Grass}(\sigma) = \mathfrak{Grass}(\mathbb{S}) \cap \text{Schu}(\sigma) \quad \text{and} \quad \text{GRASS}(\sigma) = \text{GRASS}(\mathbb{S}) \cap \text{SCHU}(\sigma). \quad \square$$

Combining Observations 3.6 and 3.7, we obtain:

**Corollary 3.8.** *The  $\mathfrak{Grass}(\sigma)$  (resp.,  $\text{GRASS}(\sigma)$ ), where  $\sigma$  runs through the skeleta in  $\hat{P}$  (resp., in  $\hat{\mathbf{P}}$ ) which are compatible with  $\mathbb{S}$ , form an open cover of  $\mathfrak{Grass}(\mathbb{S})$  (resp., of  $\text{GRASS}(\mathbb{S})$ ). In general, they fail to be open in  $\mathfrak{Grass}_{\mathbf{d}}^T$  (resp.,  $\text{GRASS}_{\mathbf{d}}^T$ ), however.  $\square$*

To describe affine coordinates for the varieties  $\mathfrak{Grass}(\sigma)$  and  $\text{GRASS}(\sigma)$ , we separate the small and large scenarios.

### 3.C. Critical paths and affine coordinates for the subvarieties $\mathfrak{Grass}(\sigma)$ of $\mathfrak{Grass}_{\mathbf{d}}^T$ .

We specialize to the situation where  $Q = P$  is the distinguished projective cover of  $T$ , keeping the notation of the two preceding sections. Moreover, we fix a skeleton  $\sigma \subseteq \hat{P}$  with dimension vector  $\mathbf{d}$  and top  $T$ . Recall that our purpose in shifting from the  $\Lambda$ -projective module  $P$  to the  $KT$ -projective module  $\hat{P}$  in defining a skeleton  $\sigma$  was to render the lengths of the paths in  $\sigma$  unambiguous. In the following supplement to Definition 3.1, this well-definedness of path lengths is once more essential.

**Definition 3.9.** A  $\sigma$ -critical path is a path of length at most  $L$  in  $\hat{P} \setminus \sigma$ , with the property that every proper initial subpath belongs to  $\sigma$ . In other words, a path in  $\hat{P}$  is  $\sigma$ -critical if and only if it fails to belong to  $\sigma$  and is of the form  $\alpha p\hat{z}_r$ , where  $\alpha$  is an arrow and  $p\hat{z}_r \in \sigma$ . Moreover, for any such  $\sigma$ -critical path  $\alpha p\hat{z}_r$ , its  $\sigma$ -set, denoted  $\sigma(\alpha p\hat{z}_r)$ , is defined to be the set of all paths  $q\hat{z}_s \in \sigma$  which are at least as long as  $\alpha p\hat{z}_r$  and end in the same vertex as  $\alpha p\hat{z}_r$ .

Finally, we let  $N$  be the disjoint union of the sets  $\{\alpha p\hat{z}_r\} \times \sigma(\alpha p\hat{z}_r)$ , where  $\alpha p\hat{z}_r$  traces the  $\sigma$ -critical paths. (We write the elements of  $N$  as pairs, since a priori, the sets  $\sigma(\alpha p\hat{z}_r) \subseteq \hat{P}$  need not be disjoint.)  $\square$

In particular, this definition entails: Whenever the length of a  $\sigma$ -critical path  $\alpha p\hat{z}_r$  exceeds the maximum of the lengths of the paths in  $\sigma$ , the corresponding  $\sigma$ -set  $\sigma(\alpha p\hat{z}_r)$

is empty. In accordance with our earlier conventions (Definition 3.1), we often identify a  $\sigma$ -critical path  $\alpha p\widehat{z}_r$  with its  $I\widehat{P}$ -residue class  $\alpha p z_r$  in  $P$ , unless we wish to emphasize well-definedness of path lengths. If  $\sigma$  is the first skeleton in Example 3.4(2), we find: The  $\sigma$ -critical paths are  $\gamma\widehat{z}_1, \gamma\alpha\widehat{z}_1, \alpha\beta\alpha\widehat{z}_1, \alpha\beta\widehat{z}_1, \beta\gamma\beta\widehat{z}_1, \beta\widehat{z}_2, \gamma\widehat{z}_2, \beta\alpha\widehat{z}_2, \gamma\alpha\widehat{z}_2, \alpha\widehat{z}_3, \beta\widehat{z}_3, \gamma\widehat{z}_3$ . The  $\sigma$ -set of  $\gamma\alpha\widehat{z}_1$ , for instance, is  $\sigma(\gamma\alpha\widehat{z}_1) = \{\beta\alpha\widehat{z}_1, \gamma\beta\widehat{z}_1, \gamma\beta\alpha\widehat{z}_1, \alpha\gamma\beta\widehat{z}_1\}$ .

First, we note that the cardinality of the set  $N$  does not actually depend on  $\sigma$ . The proof is left to the reader.

**Observation 3.10.** *Let  $\sigma$  be a skeleton with dimension vector  $\mathbf{d}$  and top  $T$ . Moreover, let  $\mathbb{S}$  be the unique semisimple sequence with which  $\sigma$  is compatible. Then the number*

$$|N| = \sum_{\alpha p\widehat{z}_r \text{ } \sigma\text{-critical}} |\sigma(\alpha p\widehat{z}_r)|$$

*depends only on  $\mathbb{S}$ , not on  $\sigma$ .  $\square$*

The next observation is obvious. We state it in order to emphasize its pivotal role.

**3.11. Key consequence of the definition.** Let  $C$  be a point in the subvariety  $\mathfrak{Grass}(\sigma)$  of  $\mathfrak{Grass}_d^T$ . Given any  $\sigma$ -critical path  $\alpha p\widehat{z}_r$ , there exist unique scalars  $c_{\alpha p\widehat{z}_r, q\widehat{z}_s}$  with the property that

$$\alpha p z_r + C = \sum_{q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)} c_{\alpha p\widehat{z}_r, q\widehat{z}_s} q z_s + C$$

in  $P/C$ . The isomorphism type of  $M = P/C$  is completely determined by the resulting family of such scalars, as  $\alpha p\widehat{z}_r$  traces all  $\sigma$ -critical paths.

Thus we obtain a map

$$\psi : \mathfrak{Grass}(\sigma) \rightarrow \mathbb{A}^N, \quad C \mapsto c = (c_{\alpha p\widehat{z}_r, q\widehat{z}_s})_{\alpha p\widehat{z}_r \text{ } \sigma\text{-critical}, q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)}. \quad \square$$

The principal aim of this section is to show that this map  $\psi$  is an isomorphism from the variety  $\mathfrak{Grass}(\sigma)$  onto a closed subvariety of  $\mathbb{A}^N$ . If we identify the paths  $p\widehat{z}_r \in \widehat{P}$  with the corresponding elements  $p z_r \in P$ , we readily find: The point  $C \in \mathfrak{Grass}(\sigma)$  corresponding to a point  $c$  in the image of  $\psi$  is the  $\Lambda$ -submodule of  $JP$  which is generated by the elements  $\alpha p z_r - \sum_{q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)} c_{\alpha p\widehat{z}_r, q\widehat{z}_s} q z_s$ , where  $\alpha p\widehat{z}_r$  runs through the  $\sigma$ -critical paths.

The following theorem generalizes the corresponding result ([8], Theorem 3.14) for  $\mathfrak{Grass}_d^T$ , where  $T$  is a *squarefree* semisimple module. It serves as a fundamental tool in [1] and [10].

**Theorem 3.12.** *Let  $\sigma \subseteq \widehat{P}$  be an abstract  $d$ -dimensional skeleton with top  $T$ .*

*The image of the map  $\psi : \mathfrak{Grass}(\sigma) \rightarrow \mathbb{A}^N$  of Observation 3.11 is a closed subvariety of  $\mathbb{A}^N$ ; in particular, it is affine.*

*Moreover,  $\psi$  is an isomorphism  $\mathfrak{Grass}(\sigma) \rightarrow \text{Im}(\psi)$ , whose inverse sends any point  $c = (c_{\alpha p\widehat{z}_r, q\widehat{z}_s})_{\alpha p\widehat{z}_r \text{ } \sigma\text{-critical}, q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)}$  to the submodule  $U(c) \subseteq JP$  which is generated by the differences*

$$\alpha p z_r - \sum_{q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)} c_{\alpha p\widehat{z}_r, q\widehat{z}_s} q z_s,$$

where  $\alpha p \widehat{z}_r$  runs through the  $\sigma$ -critical paths. This yields the following commutative diagram:

$$\begin{array}{ccc}
 \mathbb{A}^N \supseteq \text{Im}(\psi) & & \\
 \downarrow \psi^{-1} & \searrow \chi & \\
 \text{Grass}_d^T \supseteq \text{Grass}(\sigma) & \nearrow \phi & \{\text{isom. types of } \Lambda\text{-modules with skeleton } \sigma\}
 \end{array}$$

Here  $\chi$  is the map which sends any point  $c \in \text{Im}(\psi)$  to the isomorphism class of the factor module  $P/U(c)$ , and  $\phi$  the map which sends any point  $C \in \text{Grass}(\sigma)$  to the class of  $P/C$ . Polynomial equations determining the incarnation  $\psi(\text{Grass}(\sigma))$  of  $\text{Grass}(\sigma)$  in  $\mathbb{A}^N$  can be algorithmically obtained from  $\Gamma$  and any set of left ideal generators for  $I \subseteq K\Gamma$ . If  $K_0$  is a subfield of  $K$  over which left ideal generators for  $I$  are defined, the resulting polynomials determining  $\text{Grass}(\sigma)$  are defined over  $K_0$  as well.

Once the theorem is proved, we will identify points  $C \in \text{Grass}(\sigma)$  with the corresponding points  $c = \psi(C) \in \mathbb{A}^N$ , whenever convenient.

The algorithmic portion of the theorem will only be implicitly addressed here. The algorithm has been implemented in [2] with reference to the present paper.

For a proof of Theorem 3.12, we further enlarge  $K\Gamma$ , namely we consider the following (noncommutative) polynomial ring over the path algebra  $K\Gamma$ , in which the variables commute with the coefficients from  $K\Gamma$ :

$$K\Gamma[X] = K\Gamma[X_\nu \mid \nu \in N],$$

where again

$$N = \bigsqcup_{\alpha p \widehat{z}_r \text{ } \sigma\text{-critical}} \{\alpha p \widehat{z}_r\} \times \sigma(\alpha p \widehat{z}_r).$$

Moreover, we extend  $\widehat{P}$  to a projective module  $\widehat{\mathfrak{P}}$  over the enlarged base ring,

$$\widehat{\mathfrak{P}} = K\Gamma[X] \otimes_{K\Gamma} \widehat{P} = K\Gamma[X] \otimes_{K\Gamma} \left( \bigoplus_{1 \leq r \leq t} K\Gamma \widehat{z}_r \right) \cong \bigoplus_{1 \leq r \leq t} K\Gamma[X] \widehat{z}_r,$$

and consider the submodule  $\mathfrak{C}$  of  $\widehat{\mathfrak{P}}$  which is generated over  $K\Gamma[X]$  by all differences

$$\alpha p \widehat{z}_r - \sum_{q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r)} X_{\alpha p \widehat{z}_r, q \widehat{z}_s} q \widehat{z}_s,$$

where  $\alpha p \widehat{z}_r$  runs through the  $\sigma$ -critical paths. In particular,  $\alpha p \widehat{z}_r \in \mathfrak{C}$  whenever  $\sigma(\alpha p \widehat{z}_r)$  is empty. Roughly, the idea is to find polynomial equations for the set of points  $c = (c_\nu)_{\nu \in N}$

with the property that replacement of  $X_\nu$  by  $c_\nu$  turns  $\widehat{\mathfrak{P}}/\mathfrak{C}$  into a left  $\Lambda$ -module with  $K$ -basis  $\sigma$ . (Note that  $\widehat{P}$  canonically embeds into  $\widehat{\mathfrak{P}}$ , whence we may view  $\sigma$  as a subset of  $\widehat{\mathfrak{P}}$ .) It will turn out that we will, in fact, be solving a more general universal problem; see Proposition 3.15.

It will be convenient to write  $y_1 \hat{=} y_2$  whenever  $y_1, y_2 \in \widehat{\mathfrak{P}}$  have the same residue class modulo  $\mathfrak{C}$ .

**Lemma 3.13.** *The factor module  $\widehat{\mathfrak{P}}/\mathfrak{C}$  is a free module over the commutative polynomial ring  $K[X]$ , having as basis the  $\mathfrak{C}$ -residue classes of the elements in  $\sigma$ .*

*In other words, given any element  $y \in \widehat{\mathfrak{P}}$ , there exist unique polynomials  $\tau_{q\widehat{z}_s}^y \in K[X]$  such that*

$$y \hat{=} \sum_{q\widehat{z}_s \in \sigma} \tau_{q\widehat{z}_s}^y q\widehat{z}_s.$$

*Proof.* To show that, modulo  $\mathfrak{C}$ , every element  $y \in \widehat{\mathfrak{P}}$  is a  $K[X]$ -linear combination of elements in  $\sigma$ , it is clearly harmless to assume that  $y$  has the form  $u\widehat{z}_r$  for some path  $u \in K\Gamma$ ; indeed, these elements generate  $\widehat{\mathfrak{P}}$  over  $K[X]$ . Let  $u'$  be the longest initial subpath of  $u$  such that  $u'\widehat{z}_r \in \sigma$ ; this path  $u'$  may have length zero. If  $u' = u$ , then  $y \in \sigma$ . Otherwise,  $u'$  is a proper initial subpath of  $u$ , and consequently there exists a  $\sigma$ -critical initial subpath of  $u\widehat{z}_r$ , say  $\alpha u'\widehat{z}_r$ , where  $y = u''\alpha u'\widehat{z}_r$  for a suitable terminal subpath  $u''$  of  $u$ . Thus  $y \hat{=} \sum_{q\widehat{z}_s \in \sigma(\alpha u'\widehat{z}_r)} X_{\alpha u'\widehat{z}_r, q\widehat{z}_s} u''q\widehat{z}_s$ . Each of the paths  $u''q\widehat{z}_s$  occurring in this sum has an initial subpath in  $\sigma$  which is strictly longer than  $\text{length}(u')$ . Repeat the procedure with these paths. Since every  $\sigma$ -critical path  $\beta q\widehat{z}_s$  of length  $\geq L + 1$  has empty  $\sigma$ -set  $\sigma(\beta q\widehat{z}_s)$ , every path whose length exceeds  $L$  is congruent to zero. Therefore, the procedure terminates in a  $K[X]$ -linear combination of paths in  $\sigma$ .

To prove linear independence over  $K[X]$  of the residue classes  $p\widehat{z}_r + \mathfrak{C} \in \widehat{\mathfrak{P}}/\mathfrak{C}$  with  $p\widehat{z}_r \in \sigma$ , we assume the contrary. This amounts to the existence of distinct paths  $p_1\widehat{z}_{r_1}, \dots, p_m\widehat{z}_{r_m}$  in  $\sigma$ , together with nonzero elements  $a_i \in K[X]$ , such that the sum  $\sum_{1 \leq i \leq m} a_i p_i \widehat{z}_{r_i}$  belongs to  $\mathfrak{C}$ . That the listed paths are distinct in  $\widehat{\mathfrak{P}}$  means that, for  $i \neq j$ , either  $p_i \neq p_j$  or  $r_i \neq r_j$ . In light of the definition of  $\mathfrak{C}$ , our assumption translates into an equality in  $\widehat{\mathfrak{P}}$  as follows:

$$(\dagger) \quad \sum_{1 \leq i \leq m} a_i p_i \widehat{z}_{r_i} = \sum_{\alpha p \widehat{z}_r \text{ } \sigma\text{-critical}} \sum_{1 \leq j \leq m(\alpha p \widehat{z}_r)} b_{\alpha p \widehat{z}_r, j} u_{\alpha p \widehat{z}_r, j} \left( \alpha p \widehat{z}_r - \sum_{q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r)} X_{\alpha p \widehat{z}_r, q \widehat{z}_s} q \widehat{z}_s \right),$$

for suitable elements  $b_{\alpha p \widehat{z}_r, j} \in K[X]$ , distinct paths  $u_{\alpha p \widehat{z}_r, j}$  in  $\Gamma$  of lengths  $\geq 0$  starting in the endpoint of  $\alpha$ , and nonnegative integers  $m(\alpha p \widehat{z}_r)$ . Since  $\widehat{\mathfrak{P}}$  is a free  $K[X]$ -module having as basis all paths  $v\widehat{z}_r$  in the projective  $K\Gamma$ -module  $\widehat{P}$ , we are now in a position to compare coefficients. Pick a  $\sigma$ -critical path  $\beta v\widehat{z}_\mu \in \widehat{P}$  of minimal length such that some  $b_{\beta v\widehat{z}_\mu, j_0} \neq 0$ ; here  $\beta$  is an arrow and  $v\widehat{z}_\mu \in \sigma$ . Without loss of generality,  $j_0 = 1$ . Since  $w := u_{\beta v\widehat{z}_\mu, 1} \beta v\widehat{z}_\mu$  does not appear on the left-hand side of  $(\dagger)$ , it must cancel out on the



right. Observe that  $w$  does not equal any path of the form  $u_{\alpha p \hat{z}_r, j} \alpha p \hat{z}_r$  with  $\alpha p \hat{z}_r \neq \beta v \hat{z}_\mu$ , for  $v \hat{z}_\mu$  is the longest initial subpath of  $w$  which belongs to  $\sigma$ . Nor does  $w$  coincide with one of the paths  $u_{\beta v \hat{z}_\mu, j} \beta v \hat{z}_\mu$  for  $j \neq 1$ . Consequently,  $w$  must be one of the  $u_{\alpha p \hat{z}_r, j} q \hat{z}_s$  for a  $\sigma$ -critical path  $\alpha p \hat{z}_r$ , some  $q \hat{z}_s \in \sigma(\alpha p \hat{z}_r)$ , and some  $j$  with  $b_{\alpha p \hat{z}_r, j} \neq 0$ . On the other hand,

$$\text{length}(q \hat{z}_s) \geq \text{length}(\alpha p \hat{z}_r) \geq \text{length}(\beta v \hat{z}_\mu)$$

by the definition of  $\sigma(\alpha p \hat{z}_r)$  and our minimality assumption, whence  $q \hat{z}_s$  is longer than  $v \hat{z}_\mu$ . But this is absurd as, by construction,  $v \hat{z}_\mu$  is the longest initial subpath of  $w$  which belongs to  $\sigma$ . This contradiction completes the linear independence argument.  $\square$

We next describe an ideal  $\mathcal{I}(\sigma)$  of polynomials in  $K[X]$  such that  $\psi(\mathbf{Grass}(\sigma))$  is the vanishing locus of  $\mathcal{I}(\sigma)$ .

### 3.14 Polynomials for the image $\psi(\mathbf{Grass}(\sigma))$ in $\mathbb{A}^N$ .

Let  $R$  be any finite generating set for the ideal  $I \subseteq K\Gamma$  of relations for  $\Lambda$ , viewed as a left ideal of  $K\Gamma$ . Such a generating set exists, since all paths of length  $L + 1$  belong to  $I$ . It is, moreover, harmless to assume that  $R = \bigcup_{1 \leq j \leq n} Re_j$ . Consider the subset  $\hat{R}$  of  $\hat{P} \subseteq \hat{\mathfrak{P}}$ , which is defined as follows:

$$\hat{R} = \{\rho \hat{z}_r \mid \rho \in R, 1 \leq r \leq t, \rho e(r) = \rho\}.$$

Again,  $e(r)$  is the primitive idempotent norming  $\hat{z}_r$ , that is,  $\hat{z}_r$  is a path of length zero in  $\hat{P}$ , which starts and ends in  $e(r)$ . Recall that, for any  $i \in \{1, \dots, n\}$ , the number of indices  $r$  with  $e(r) = e_i$  equals the multiplicity  $t_i$  of  $S_i$  in  $T$ . Hence, certain of the relations in  $R$  will “fan out” to “multiple incarnations” in  $\hat{R}$ , while the elements  $\rho \in R$  with  $\rho e_j = \rho$  and  $S_j$  not a summand of  $T$  will not make an appearance in  $\hat{R}$ .

By Lemma 3.13, there exist, for each  $\rho \hat{z}_r \in \hat{R}$ , unique polynomials  $\tau_{q \hat{z}_s}^{\rho \hat{z}_r}(X) \in K[X]$  with the property that  $\rho \hat{z}_r \hat{= \sum_{q \hat{z}_s \in \sigma} \tau_{q \hat{z}_s}^{\rho \hat{z}_r}(X) q \hat{z}_s}$ . Define  $\mathcal{I}(\sigma) \subset K[X]$  to be the ideal generated by all the  $\tau_{q \hat{z}_s}^{\rho \hat{z}_r}$ , where  $\rho \hat{z}_r$  traces  $\hat{R}$ .

It is readily seen that the ideal  $\mathcal{I}(\sigma)$  in  $K[X]$  does not depend on our specific choice of  $R$ , a fact which will also emerge as a consequence of Proposition 3.15.

Essentially, our proof of Theorem 3.12 rests on the fact that  $\psi(\mathbf{Grass}(\sigma))$  equals the vanishing set of the ideal  $\mathcal{I}(\sigma)$  in  $\mathbb{A}^N$ . As a by-product, we find that  $\mathcal{I}(\sigma)$  in fact plays a universal role relative to path algebras based on  $\Gamma$  and  $I$ , with coefficients in an arbitrary commutative  $K$ -algebra.

Indeed, our setup can be generalized as follows: Letting  $\mathcal{A}$  be any commutative  $K$ -algebra, we consider the path algebra (resp., path algebra modulo relations)  $\mathcal{A}\Gamma \cong \mathcal{A} \otimes_K K\Gamma$  (resp.,  $\mathcal{A}\Gamma/\mathcal{A}I \cong \mathcal{A} \otimes_K \Lambda$ ) with coefficients in  $\mathcal{A}$ . The projective  $\mathcal{A}\Gamma$ -module  $\hat{P}(\mathcal{A}) = \bigoplus_{1 \leq r \leq t} \mathcal{A}\Gamma \hat{z}_r$ , where the  $\hat{z}_r$  are again top elements normed by the vertices  $e(r)$  in the quiver  $\Gamma$ , is free as an  $\mathcal{A}$ -module, having as basis the set of all paths in  $\hat{P}$  (as introduced ahead of Definition 3.1). It specializes to  $\hat{\mathfrak{P}}$  for  $\mathcal{A} = K[X]$ , and to  $\hat{P}$  for  $\mathcal{A} = K$ . In particular,  $\hat{P} = \hat{P}(K) \subseteq \hat{P}(\mathcal{A})$ , so that  $\sigma$  can also be considered as a subset of  $\hat{P}(\mathcal{A})$ .

Finally, given an element  $\mathfrak{a} = (\mathfrak{a}_\nu)_{\nu \in N} \in \mathcal{A}^N$ , we let  $U(\mathfrak{a})$  be the submodule of  $\widehat{P}(\mathcal{A})$  generated by the differences

$$\alpha p\widehat{z}_r - \sum_{q\widehat{z}_s \in \sigma(\alpha p\widehat{z}_r)} \mathfrak{a}_{\alpha p\widehat{z}_r, q\widehat{z}_s} q\widehat{z}_s,$$

where  $\alpha p\widehat{z}_r$  runs through the  $\sigma$ -critical paths in  $\widehat{P}$ , identified with the corresponding elements in  $\widehat{P}(\mathcal{A})$ .

**Proposition 3.15.** *The  $K$ -algebra  $\mathcal{A}_0 = K[X]/\mathcal{I}(\sigma)$  has the following universal property relative to the skeleton  $\sigma$ . Given any  $K$ -algebra homomorphism  $\eta$  from  $\mathcal{A}_0$  to another commutative  $K$ -algebra  $\mathcal{A}$ , set  $\mathfrak{a} = (\eta(\overline{X}_\nu)) \in \mathcal{A}^N$ . Then the factor module*

$$M = \widehat{P}(\mathcal{A})/U(\mathfrak{a})$$

*is an  $\mathcal{A} \otimes_K \Lambda$ -module with the following property: For  $0 \leq l \leq L$ , the layer  $\mathcal{J}^l M / \mathcal{J}^{l+1} M$  is a free  $\mathcal{A}$ -module having as basis the residue classes of the  $p\widehat{z}_r$  in  $\sigma_l$ ; here  $\sigma_l$  is the set of paths of length  $l$  in  $\sigma$  as before, and  $\mathcal{J}$  denotes the ideal  $\mathcal{A} \otimes_K J$  in  $\mathcal{A} \otimes_K \Lambda$ .*

*Conversely: If  $\mathcal{A}$  is a commutative  $K$ -algebra and  $M = \widehat{P}(\mathcal{A})/U$  is an  $\mathcal{A} \otimes_K \Lambda$ -module satisfying the above layer condition, then there exists a unique  $K$ -algebra homomorphism  $\eta : \mathcal{A}_0 \rightarrow \mathcal{A}$  such that  $U = U(\eta(\overline{X}_\nu))$ .*

*Proof.* Let  $\eta$  and  $\mathfrak{a}$  be as in the first assertion. Then  $\mathfrak{a} = (\mathfrak{a}_\nu)_{\nu \in N}$  belongs to the vanishing set  $V_{\mathcal{A}}(\mathcal{I}(\sigma))$  of  $\mathcal{I}(\sigma)$  in  $\mathcal{A}^N$ , i.e.,  $\tau(\mathfrak{a}) = 0$  for all  $\tau \in \mathcal{I}(\sigma)$ . The definition of  $\mathcal{I}(\sigma)$  entails that  $M = \widehat{P}(\mathcal{A})/U(\mathfrak{a})$  is annihilated by  $I$ , which makes  $M$  an  $\mathcal{A} \otimes_K \Lambda$ -module. In particular,  $\mathcal{J}^{L+1} M = 0$ .

In a first step, we show that  $M$  is a free  $\mathcal{A}$ -module having basis  $p\widehat{z}_r + U(\mathfrak{a})$ , where  $p\widehat{z}_r$  runs through  $\sigma$ . Preceding  $\eta$  by the quotient map  $K[X] \rightarrow \mathcal{A}_0$  yields a  $K$ -algebra homomorphism  $\overline{\eta} : K[X] \rightarrow \mathcal{A}$ , which in turn induces an evaluation map  $\theta : \widehat{\mathfrak{P}} = \widehat{P}(K[X]) \rightarrow \widehat{P}(\mathcal{A})$  from the free  $K[X]$ -module  $\widehat{\mathfrak{P}}$  to the free  $\mathcal{A}$ -module  $\widehat{P}(\mathcal{A})$ . It is semilinear in the sense that  $\theta(\tau y) = \overline{\eta}(\tau)\theta(y)$  for  $y \in \widehat{\mathfrak{P}}$  and  $\tau \in K[X]$ . When  $\mathcal{A}$  is endowed with the  $K[X]$ -module structure induced by  $\overline{\eta}$ , the map  $\theta$  gives rise to an isomorphism of free  $\mathcal{A}$ -modules

$$\mathcal{A} \otimes_{K[X]} \widehat{\mathfrak{P}} \rightarrow \widehat{P}(\mathcal{A}), \quad 1 \otimes q\widehat{z}_s \mapsto q\widehat{z}_s,$$

which maps the preferred path basis of  $\mathcal{A} \otimes_{K[X]} \widehat{\mathfrak{P}}$  to that of  $\widehat{P}(\mathcal{A})$ ; here  $q\widehat{z}_s$  traces the paths in  $\widehat{P}$ . By construction,  $\theta$  takes the defining generators for the submodule  $\mathfrak{C} \subseteq \widehat{\mathfrak{P}}$  to the defining generators for the submodule  $U(\mathfrak{a}) \subseteq \widehat{P}(\mathcal{A})$ , and by Lemma 3.13,  $\widehat{\mathfrak{P}}/\mathfrak{C}$  is still free over  $K[X]$ , namely on basis  $p\widehat{z}_r + \mathfrak{C}$ , where  $p\widehat{z}_r$  traces  $\sigma$ . Thus we obtain an isomorphism  $\mathcal{A} \otimes_{K[X]} (\widehat{\mathfrak{P}}/\mathfrak{C}) \cong \widehat{P}(\mathcal{A})/U(\mathfrak{a})$  sending  $p\widehat{z}_r + \mathfrak{C}$  to  $p\widehat{z}_r + U(\mathfrak{a})$  for all  $p\widehat{z}_r \in \sigma$ . We infer that the latter residue classes,  $p\widehat{z}_r + U(\mathfrak{a})$ , indeed form a basis for  $\widehat{P}(\mathcal{A})/U(\mathfrak{a})$  over  $\mathcal{A}$ .

To establish the first claim of the proposition, it now suffices to prove that, for  $0 \leq l \leq L$ , the  $p\widehat{z}_r$  in  $\sigma_l$  are  $\mathcal{A}$ -linearly independent modulo  $\mathcal{J}^{l+1} M$ . Assuming this to be false, we

let  $l_0$  be minimal with respect to failure. Then  $l_0 \geq 1$ , and it is harmless to assume that  $l_0 = L$ . For otherwise, we may enlarge the ideal  $I \subseteq K\Gamma$  so that it contains the paths of length  $l_0 + 1$  next to the original relations, replace  $\sigma$  by  $\bigcup_{0 \leq l \leq l_0} \sigma_l$ , adjust the ideal  $\mathcal{I}(\sigma) \subseteq K[X]$  according to 3.14, and replace  $M$  by  $M/\mathcal{J}^{l_0+1}M$ . In this modified setup,  $l_0$  is still minimal with respect to failure of our independence condition. So we only need to refute the assumption that the elements  $p\hat{z}_r + U(\mathbf{a})$ ,  $p\hat{z}_r \in \sigma_L$ , are  $\mathcal{A}$ -linearly independent in  $M$ . But this was already established in the first paragraph.

For the second claim, let  $U$  be an  $(\mathcal{A} \otimes_K K\Gamma)$ -submodule of  $\hat{P}(\mathcal{A})$  such that  $M = \hat{P}(\mathcal{A})/U$  is an  $(\mathcal{A} \otimes_K \Lambda)$ -module with the described layer property. In particular,  $M$  is then a free  $\mathcal{A}$ -module with basis  $p\hat{z}_r + U$ , where  $p\hat{z}_r$  traces  $\sigma$ . Letting  $\alpha p\hat{z}_r$  be any  $\sigma$ -critical path of length  $l$ , our hypothesis on the layers of  $M$  yields coefficients  $\mathbf{b}_{\alpha p\hat{z}_r, q\hat{z}_s} \in \mathcal{A}$ , for  $q\hat{z}_s \in \sigma_l$ , such that

$$\left( \alpha p\hat{z}_r - \sum_{q\hat{z}_s \in \sigma_l} \mathbf{b}_{\alpha p\hat{z}_r, q\hat{z}_s} q\hat{z}_s \right) + U \in \mathcal{J}^{l+1}M.$$

Clearly, we may assume that the above sum involves only paths ending in the same vertex as  $\alpha p\hat{z}_r$ , meaning that all of the  $q\hat{z}_s$ 's making an appearance belong to  $\sigma(\alpha p\hat{z}_r)$ . An induction on  $l \leq L$  thus yields elements  $\mathbf{a}_\nu \in \mathcal{A}$ , for  $\nu \in N$ , such that

$$(\dagger) \quad \alpha p\hat{z}_r - \sum_{q\hat{z}_s \in \sigma(\alpha p\hat{z}_r)} \mathbf{a}_{\alpha p\hat{z}_r, q\hat{z}_s} q\hat{z}_s \in U$$

for any  $\sigma$ -critical path  $\alpha p\hat{z}_r$ . Uniqueness of the point  $\mathbf{a} = (\mathbf{a}_\nu)_{\nu \in N} \in \mathcal{A}^N$  follows from freeness of  $M$  over  $\mathcal{A}$ . This provides us with a unique  $K$ -algebra homomorphism  $\hat{\eta} : K[X] \rightarrow \mathcal{A}$  such that  $(\hat{\eta}(X_\nu)) = \mathbf{a}$ . By construction, the map  $\theta : \hat{\mathfrak{P}} = \hat{P}(K[X]) \rightarrow \hat{P}(\mathcal{A})$  induced by  $\hat{\eta}$  sends the generators of the submodule  $\mathfrak{C} \subseteq \hat{\mathfrak{P}}$  to the differences displayed in  $(\dagger)$ , so  $\theta(\mathfrak{C}) \subseteq U$ . It now follows from Lemma 3.13 that  $\theta(y) - \sum_{q\hat{z}_s \in \sigma} \tau_{q\hat{z}_s}^y(\mathbf{a}) q\hat{z}_s \in U$  for all  $y \in \hat{\mathfrak{P}}$ . Since  $I\hat{P}(\mathcal{A}) \subseteq U$  (because  $M$  is an  $\mathcal{A} \otimes_K \Lambda$ -module), this yields  $\sum_{q\hat{z}_s \in \sigma} \tau_{q\hat{z}_s}^{\rho\hat{z}_r}(\mathbf{a}) q\hat{z}_s \in U$  for all  $\rho\hat{z}_r \in \hat{R}$ . Linear independence of the cosets  $q\hat{z}_s + U$  thus implies  $\mathbf{a} \in V_{\mathcal{A}}(\mathcal{I}(\sigma))$ . Therefore  $\hat{\eta}$  induces a  $K$ -algebra homomorphism  $\eta : \mathcal{A}_0 \rightarrow \mathcal{A}$  such that  $(\eta(\overline{X}_\nu)) = \mathbf{a}$ . By construction of  $\mathbf{a}$ , the submodule  $U(\mathbf{a})$  of  $\hat{P}(\mathcal{A})$  is contained in  $U$ . But as we saw in the proof of the first claim,  $\hat{P}(\mathcal{A})/U(\mathbf{a})$  is free of rank  $|\sigma|$  over  $\mathcal{A}$ ; this rank coincides with the free rank of  $\hat{P}(\mathcal{A})/U$  by hypothesis. We conclude  $U = U(\mathbf{a})$  as required. Finally, uniqueness of  $\eta$  holds by the uniqueness of  $\mathbf{a}$ .  $\square$

*Proof of Theorem 3.12.* Specializing Proposition 3.15 to  $\mathcal{A} = K$ , we find that the image of the map  $\psi$  equals the vanishing set  $V(\mathcal{I}(\sigma))$  of the ideal  $\mathcal{I}(\sigma) \subseteq K[X]$  in  $\mathbb{A}^N = \mathbb{A}^N(K)$ , and that the inverse  $\psi^{-1}$  sends any point  $c \in V(\mathcal{I}(\sigma))$  to the submodule  $C = U(c)$  of  $\hat{P}$ . That  $\psi$  and  $\psi^{-1}$  are morphisms of varieties follows from an argument analogous to that of [3], Theorem A, an early precursor of Theorem 3.12. Thus  $\psi$  is indeed an isomorphism.

That the diagram in the statement of the theorem is commutative is now clear. Suppose that  $K_0$  is a subfield of  $K$  with the property that elements generating  $I \subseteq K\Gamma$  as a left ideal can be found in  $K_0\Gamma$ . Since the construction of the generators  $\rho_{q\hat{z}_s}^{\alpha p\hat{z}_r}$  for  $\mathcal{I}(\sigma)$  described

in 3.14 may be carried out by applying Lemma 3.13 to the  $K_0$ -algebra  $K_0\Gamma/(I \cap K_0\Gamma)$ , these polynomials may be assumed to belong to  $K_0[X]$ . As already mentioned, the claim concerning the algorithmic nature is backed by the code in [2].  $\square$

### 3.D. Affine coordinates for the subvarieties $\text{GRASS}(\sigma)$ of $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

In this section, we outline how the definitions and methods of the preceding section can be adapted to the case where the distinguished projective  $\Lambda$ -module considered is  $\mathbf{P} = \bigoplus_{1 \leq r \leq d} \Lambda z_r$  whose top  $\mathbf{P}/J\mathbf{P}$  equals the semisimple left  $\Lambda$ -module of dimension vector  $\mathbf{d}$ . As before,  $\hat{\mathbf{P}} = \bigoplus_{1 \leq r \leq d} K\Gamma \hat{z}_r$  is the corresponding projective  $K\Gamma$ -module, and each top element  $z_r$  of  $\mathbf{P}$  or  $\hat{z}_r$  of  $\hat{\mathbf{P}}$  is normed by the primitive idempotent  $e(r)$ . Moreover, we assume that  $z_r$  coincides with the image of  $\hat{z}_r$  under the canonical epimorphism  $\hat{\mathbf{P}} \rightarrow \mathbf{P} = \hat{\mathbf{P}}/I\hat{\mathbf{P}}$ .

Again, we fix an abstract skeleton  $\sigma$  with dimension vector  $\mathbf{d}$  and top  $T$ , but this time, we choose  $\sigma$  as a subset of  $\hat{\mathbf{P}}$ . Recall that the dimension vector of  $T$  is  $\mathbf{t} = (t_1, \dots, t_n)$ , whence  $t := \sum_i t_i$  is the  $K$ -dimension of  $T$ . Thus, it is clearly harmless to assume that the paths of length zero in  $\sigma$  are  $\hat{z}_1, \dots, \hat{z}_t$ , and to identify the distinguished projective  $K\Gamma$ -module  $\hat{P}$  covering  $T$  with the direct summand  $\bigoplus_{1 \leq r \leq t} K\Gamma \hat{z}_r$  of  $\hat{\mathbf{P}}$ . Next we define  $\sigma$ -criticality; we do so by carrying over the first of the equivalent descriptions of a  $\sigma$ -critical path in Section 3.C to the present situation. We restate the definition for emphasis.

**Definition 3.16 (pendant to Definition 3.9) and Comments.** A  $\sigma$ -critical path is a path  $q\hat{z}_r$  in  $\hat{\mathbf{P}} \setminus \sigma$ , of length at most  $L$ , with the property that every proper initial subpath of  $q\hat{z}_r$  belongs to  $\sigma$ .

Note that we obtain two types of  $\sigma$ -critical paths in the enlarged scenario: The  $\sigma$ -critical paths in  $\hat{P}$ ; these are precisely the  $\sigma$ -critical paths of positive length having the form  $\alpha p\hat{z}_r$ , where  $\alpha$  is an arrow and  $p\hat{z}_r \in \sigma$ . In addition, we have the  $\sigma$ -critical paths in the complementary summand  $\bigoplus_{t+1 \leq r \leq d} K\Gamma \hat{z}_r$ ; these are precisely the paths  $\hat{z}_{t+1}, \dots, \hat{z}_d$  of length zero in  $\hat{\mathbf{P}} \setminus \sigma$ . Thus every  $\sigma$ -critical path is of the form  $u\hat{z}_r$ , where  $u$  now is a path of length  $\geq 0$  in  $K\Gamma$ .

Given a  $\sigma$ -critical path  $u\hat{z}_r$ , its  $\sigma$ -set,  $\sigma(u\hat{z}_r)$ , consists of the paths  $q\hat{z}_s \in \sigma$  of length  $\geq \text{length}(u)$  which end in the same vertex as  $u$ . So, in contrast to the situation of Section 3.C, the set  $\sigma(u\hat{z}_r)$  may contain paths of length zero unless  $d_i = t_i$  whenever  $t_i \neq 0$ . In fact, each element  $\hat{z}_r \in \hat{P}$  with  $t < r \leq d$  is  $\sigma$ -critical, and  $\sigma(\hat{z}_r)$  contains all those candidates  $\hat{z}_s$  among  $\hat{z}_1, \dots, \hat{z}_t$  for which  $e(r) = e(s)$ .

As in Section 3.C, we denote by  $N$  the set of all pairs  $(u\hat{z}_r, q\hat{z}_s)$  such that the first entry is a  $\sigma$ -critical path and the second entry belongs to the corresponding  $\sigma$ -set  $\sigma(u\hat{z}_r)$ . Since the critical paths of positive length play a different role from those of length zero, we split up the index set accordingly:  $N = N_1 \sqcup N_0$ , where

$$N_1 = \{(\alpha p\hat{z}_r, q\hat{z}_s) \mid \alpha p\hat{z}_r \text{ } \sigma\text{-critical of positive length, } q\hat{z}_s \in \sigma(\alpha p\hat{z}_r)\}$$

and

$$N_0 = \{(\hat{z}_r, q\hat{z}_s) \mid \hat{z}_r \text{ } \sigma\text{-critical of length 0, } q\hat{z}_s \in \sigma(\hat{z}_r)\}.$$

Refer to Examples 3.5 for illustration.

To obtain a family of scalars in  $\mathbb{A}^N$  pinning down the isomorphism type of  $\mathbf{P}/C$  for a point  $C \in \text{GRASS}(\sigma)$ , we observe as in 3.11: For any  $\sigma$ -critical path  $u\widehat{z}_r$ , there exist unique scalars  $c_{u\widehat{z}_r, q\widehat{z}_s} \in K$  such that

$$uz_r + C = \left( \sum_{q\widehat{z}_s \in \sigma(u\widehat{z}_r)} c_{u\widehat{z}_r, q\widehat{z}_s} qz_s \right) + C$$

in  $\mathbf{P}/C$ . The family of scalars  $(c_\nu)_{\nu \in N}$  thus obtained, as  $u\widehat{z}_r$  traces the  $\sigma$ -critical paths, determines  $\mathbf{P}/C$  up to isomorphism; more strongly, it pins down  $C$  (see Theorem 3.17 below). As a consequence, we again obtain a morphism of varieties

$$\Psi : \text{GRASS}(\sigma) \rightarrow \mathbb{A}^N, \quad C \mapsto (c_\nu)_{\nu \in N}.$$

The first part of the following sibling of Theorem 3.12 relates the big varieties  $\text{GRASS}(\sigma)$  to the small versions.

**Theorem 3.17 (pendant to Theorem 3.12).** *Let  $\sigma \subseteq \widehat{\mathbf{P}}$  be a skeleton with top  $T$  and dimension vector  $\mathbf{d}$ .*

- (1) *The variety  $\text{GRASS}(\sigma)$  is isomorphic to  $\mathfrak{GRASS}(\sigma) \times \mathbb{A}^{N_0}$ . In particular,  $\text{GRASS}(\sigma)$  is an affine variety that differs from  $\mathfrak{GRASS}(\sigma)$  only by a direct factor which is a full affine space.*
- (2) *The map  $\Psi : \text{GRASS}(\sigma) \rightarrow \mathbb{A}^N$  induces an isomorphism from  $\text{GRASS}(\sigma)$  onto a closed subvariety of  $\mathbb{A}^N$ . Its inverse  $\Psi^{-1}$  sends any point  $c = (c_\nu)_{\nu \in N}$  to the submodule  $U(c) \subseteq \mathbf{P}$  which is generated by the differences*

$$uz_r - \sum_{q\widehat{z}_s \in \sigma(u\widehat{z}_r)} c_{u\widehat{z}_r, q\widehat{z}_s} qz_s,$$

where  $u\widehat{z}_r$  runs through the  $\sigma$ -critical paths. This isomorphism makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{A}^N \supseteq \text{Im}(\Psi) & & \\ \downarrow \Psi^{-1} & \searrow \chi & \\ \text{GRASS}_{\mathbf{d}}(\Lambda) \supseteq \text{GRASS}(\sigma) & \nearrow \phi & \{\text{isom. types of } \Lambda\text{-modules with skeleton } \sigma\} \end{array}$$

Here  $\chi$  is the map which sends any point  $c$  in  $\text{Im}(\psi)$  to the isomorphism class of  $\mathbf{P}/U(c)$ , and  $\phi$  the map which sends any point  $C \in \text{GRASS}(\sigma)$  to the class of  $\mathbf{P}/C$ .

- (3) *If  $K_0$  is a subfield of  $K$  such that generators for the left ideal  $I$  can be found in  $K_0\Gamma$ , then polynomials defining  $\Psi(\text{GRASS}(\sigma))$  in  $\mathbb{A}^N$  can be chosen in  $K_0[X]$ .*

In light of Theorem 3.12, this “big” version of the affine cover theorem is immediate from the following lemma. Since the proof of the latter is straightforward, we leave it to the reader.

**Lemma 3.18.** *We let  $\sigma$  be a skeleton as in Theorem 3.17. Retaining the preceding notation, we suppose that  $\widehat{z}_1, \dots, \widehat{z}_t$  are the paths of length zero in  $\sigma$  and view the direct summand  $P = \bigoplus_{1 \leq r \leq t} \Lambda z_r$  of  $\mathbf{P}$  as the distinguished  $\Lambda$ -projective cover of  $T$ .*

(1) *Each point  $C \in \text{GRASS}(\sigma)$  has a  $\Lambda$ -module decomposition  $C = C_1 \oplus C_0$ , where*

$$C_1 = \sum_{\alpha p \widehat{z}_r \text{ } \sigma\text{-critical of positive length}} \Lambda \left( \alpha p z_r - \sum_{q \widehat{z}_s \in \sigma(\alpha p \widehat{z}_r)} c_{\alpha p \widehat{z}_r, q \widehat{z}_s} q z_s \right)$$

and

$$C_0 = \sum_{t+1 \leq r \leq d} \Lambda \left( z_r - \sum_{q \widehat{z}_s \in \sigma(\widehat{z}_r)} c_{\widehat{z}_r, q \widehat{z}_s} q z_s \right)$$

for some point  $c \in \mathbb{A}^N$ . In particular,  $C = U(c)$ .

(2)  $C_1 = C \cap P$  belongs to  $\mathfrak{Grass}(\sigma)$ , the affine subvariety of  $\mathfrak{Grass}_d^T$  which is determined by the skeleton  $\sigma$ , when the latter is viewed as a subset of  $\widehat{P}$ . Moreover,  $C_0$  is isomorphic to  $\bigoplus_{t+1 \leq r \leq d} \Lambda z_r$ , and hence is a complement of  $P$  in  $\mathbf{P}$ . Both  $C_1$  and  $C_0$  are uniquely determined by  $C$ .

(3) Conversely, given any element  $C_1 \in \mathfrak{Grass}(\sigma) \subseteq \mathfrak{Grass}_d^T$  and any element  $(c_\nu)_{\nu \in N_0} \in \mathbb{A}^{N_0}$ , let  $C_0$  be the  $\Lambda$ -submodule of  $\mathbf{P}$  which is generated by the differences

$$z_r - \sum_{q \widehat{z}_s \in \sigma(\widehat{z}_r)} c_{\widehat{z}_r, q \widehat{z}_s} q z_s$$

with  $(\widehat{z}_r, q \widehat{z}_s)$  running through  $N_0$ . Then  $\mathbf{P} = P \oplus C_0$ , and  $C_1 \oplus C_0$  belongs to  $\text{GRASS}(\sigma)$ .  $\square$

#### 4. DEGENERATIONS IN THE THE GRASSMANNIAN MODULE VARIETIES

One of the main areas of application of the Grassmannian module varieties lies in the fact that they permit a useful alternate perspective on degenerations. Recall that a *degeneration* of a module  $M \in \Lambda\text{-mod}$  of dimension vector  $\mathbf{d}$ , which is represented by a point  $x \in \mathbf{Mod}_{\mathbf{d}}(\Lambda)$  say, is any module represented by a point in the  $GL_d$ -orbit closure of  $x$  in  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$ . As is well known, the relation  $(M \leq M' \iff M \text{ degenerates to } M')$  defines a partial order on the set of isomorphism types of modules of a fixed dimension vector. A degeneration  $M'$  of  $M$  is called *top-stable* (resp., *layer-stable*) in case  $M'/JM' = M/JM$  (resp.,  $\mathbb{S}(M') = \mathbb{S}(M)$ ). It is well-known (see [13]) that any point  $x'$  in the closure of  $GL_d.x$  in  $\mathbf{Mod}_{\mathbf{d}}(\Lambda)$  is connected to  $x$  by way of a rational curve. This statement can be improved in the projective setting, due to the following result of Kollár ([9], Proposition 3.6): Whenever  $V$  is a unirational irreducible projective variety over  $K$  (meaning that the function field of  $V$  embeds into a purely transcendental extension field of  $K$ ), any two points in  $V$  are linked by a curve isomorphic to  $\mathbb{P}^1$ ; in other words, given  $C$  and  $C'$  in  $V$ , there exists a morphism  $\phi : \mathbb{P}^1 \rightarrow V$ , such that  $\text{Im}(\phi)$  contains both  $C$  and  $C'$ . Observe that all our acting automorphism groups are rational varieties, whence the orbits in the considered Grassmannians under the actions of the pertinent automorphism groups of projectives are all unirational, as are their (projective) closures. Combining Propositions 2.1 and 2.5 with curve connectedness of the orbit closures, we obtain:

**Proposition 4.1.** *Let  $M \in \Lambda\text{-mod}$  have top  $T$  and dimension vector  $\mathbf{d}$ . Moreover, let  $\mathbf{P}$  be the distinguished projective cover of the semisimple module with dimension vector  $\mathbf{d}$ , and  $Q$  a projective module with  $T \leq Q/JQ \leq \mathbf{P}/J\mathbf{P}$ , such that  $Q$  is in turn equipped with a distinguished sequence of top elements, as in Section 2.D.*

- (a) *If  $M = Q/C$  with  $C \in [\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ , the following statements are equivalent:*
- *$M'$  is a degeneration of  $M$  with  $M'/JM' \leq Q/JQ$ ;*
  - *$M' \cong Q/C'$ , where  $C'$  belongs to the closure of  $\text{Aut}_{\Lambda}(Q).C$  in  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ ;*
  - *$M' \cong Q/C'$ , where  $C$  and  $C'$  belong to the image of a curve  $\mathbb{P}^1 \rightarrow \overline{\text{Aut}_{\Lambda}(Q).C}$ , the latter being the closure of  $\text{Aut}_{\Lambda}(Q).C$  in  $[\text{GRASS}_{\mathbf{d}}(\Lambda)]_Q$ .*
- (b) *If  $M = \mathbf{P}/C$  with  $C \in \text{GRASS}_{\mathbf{d}}(\Lambda)$ , the following statements are equivalent:*
- *$M'$  is a degeneration of  $M$ ;*
  - *$M' \cong \mathbf{P}/C'$ , where  $C'$  belongs to the closure of  $\text{Aut}_{\Lambda}(\mathbf{P}).C$  in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ ;*
  - *$M' \cong \mathbf{P}/C'$ , where  $C$  and  $C'$  belong to the image of a curve  $\mathbb{P}^1 \rightarrow \overline{\text{Aut}_{\Lambda}(\mathbf{P}).C}$ , the latter being the closure of  $\text{Aut}_{\Lambda}(\mathbf{P}).C$  in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .  $\square$*

In the special case where  $Q$  coincides with the projective cover  $P$  of  $T$ , the equivalences under (a) have already been used to advantage to explore top-stable degenerations of  $M$ , both theoretically and computationally (see [9]).

There is a special class of degenerations which is particularly accessible by way of the Grassmannian varieties:

**Definition 4.2.** A degeneration  $M' \cong \mathbf{P}/C'$  of  $M \cong \mathbf{P}/C$  is called *unipotent* in case  $C'$  belongs to the closure of  $(\text{Aut}_{\Lambda}(\mathbf{P}))_u.C$  in  $\text{GRASS}_{\mathbf{d}}(\Lambda)$ .

In Proposition 4.4. and Corollary 4.5, we will see that, in exploring unipotent degenerations, we may again cut down to smaller settings, depending on the bound we place on the size of the tops.

As a first application of Rosenlicht's result on unipotent group actions (Theorem 2.14(2)), we find that no module has a proper layer-stable degeneration which is unipotent.

**Theorem 4.3.** *If  $M, M' \in \Lambda\text{-mod}$  such that  $M'$  is a proper unipotent degeneration of  $M$ , then  $\mathbb{S}(M')$  properly dominates  $\mathbb{S}(M)$  in the sense of Definition 2.10. In particular,  $\mathbb{S}(M') \neq \mathbb{S}(M)$ .*

*Proof.* Suppose that  $M \cong \mathbf{P}/C$  and  $M' \cong \mathbf{P}/C'$  with  $C, C' \in \text{GRASS}_{\mathbf{d}}(\Lambda)$ . If  $\mathbb{S}(M) = \mathbb{S}$ , then  $(\text{Aut}_{\Lambda}(\mathbf{P}))_u.C$  is a closed subvariety of  $\text{GRASS}(\mathbb{S})$  by Theorem 2.14(2): Indeed, the  $\text{GRASS}(\sigma)$ , for  $\sigma$  compatible with  $\mathbb{S}$ , form an affine open cover of  $\text{GRASS}(\mathbb{S})$  which is stable under the  $(\text{Aut}_{\Lambda}(\mathbf{P}))_u$ -action, whence the intesection  $\text{GRASS}(\sigma) \cap ((\text{Aut}_{\Lambda}(\mathbf{P}))_u.C)$  is closed for each  $\sigma$ . Since  $C' \notin \text{Aut}_{\Lambda}(\mathbf{P}).C$ , and a fortiori  $C' \notin (\text{Aut}_{\Lambda}(\mathbf{P}))_u.C$ , we therefore conclude  $C' \notin \text{GRASS}(\mathbb{S})$ . Consequently,  $C' \in \text{GRASS}\mathbb{S}'$  for some semisimple sequence  $\mathbb{S}'$  which properly dominates  $\mathbb{S}$  by Observation 2.11.  $\square$

As a consequence of Theorem 4.3, we find that the layer-stable degenerations of  $M$  in ([9], Example 5.8) are not unipotent. On the other hand, usually, there is a plethora of unipotent degenerations (cf. Example 4.6).

We next apply Kollár's result to the closure of  $(\mathrm{Aut}_\Lambda(Q))_u.C$  in  $[\mathrm{GRASS}_d(\Lambda)]_Q$ . The geometric structure of such an orbit being well understood by Theorem 2.14(1), curve connectedness takes on a particularly user-friendly form. It will be convenient to return to the limit notation for extensions of nonsingular rational curves in an irreducible projective variety, which was used in [9], Proposition 4.2 and Corollary 5.4: Namely, if  $U$  is a dense open subset of  $K = \mathbb{A}^1 \subset \mathbb{P}^1$ , and  $\rho : U \rightarrow \overline{(\mathrm{Aut}_\Lambda(Q))_u.C}$  a morphism, then  $\rho$  extends uniquely to a curve  $\bar{\rho}$  defined on  $U \cup \{\infty\}$ ; hence, it is unambiguous to write  $\lim_{\tau \rightarrow \infty} \rho(\tau)$  for  $\bar{\rho}(\infty)$ .

**Proposition 4.4.** *Let  $C \in [\mathrm{GRASS}_d(\Lambda)]_Q$ . Moreover, for each of the distinguished top elements  $z_r$  of  $Q$ , pick finitely many paths  $p_{ri}z_{j(r,i)}$  in  $JQ$  such that, modulo  $\mathrm{Hom}_\Lambda(Q, C \cap JQ)$ , each map in the space  $\mathrm{Hom}_\Lambda(Q, JQ)$  is of the form  $z_r \mapsto \sum_i a_{ri} p_{ri} z_{j(r,i)}$  for suitable  $a_{ri} \in K$ ,  $1 \leq r \leq m$ . (In particular, for each  $r$ , we only need to consider paths  $p_{ri}$  that end in  $e(r)$ .)*

*Then the unipotent degenerations of  $Q/C$  with  $\mathrm{top} \leq Q/JQ$  are precisely the quotients  $Q/C'$  with*

$$C' = \lim_{\tau \rightarrow \infty} g_\tau(C), \text{ where } g_\tau(z_r) = z_r + \sum_i a_{ri}(\tau) p_{ri} z_{j(r,i)};$$

*here the  $a_{ri}(\tau)$  trace the rational functions in  $K(\tau)$ .*

*Proof.* First we consider the case  $Q = \mathbf{P}$ . Say  $\mathbf{P}/C'$  is a unipotent degeneration of  $\mathbf{P}/C$ . This means that both  $C$  and  $C'$  belong to the projective variety  $\overline{(\mathrm{Aut}_\Lambda(\mathbf{P}))_u.C}$ , whence, by Kollár's result, there exists a curve  $\phi : \mathbb{P}^1 \rightarrow \overline{(\mathrm{Aut}_\Lambda(\mathbf{P}))_u.C}$  such that  $C' = \lim_{\tau \rightarrow \infty} \phi(\tau)$ . In light of the comments preceding Proposition 2.14,  $\phi$  restricts to a curve  $\tau \mapsto g_\tau(C)$  of the indicated format, with  $g_\tau(z_r) = z_r + \sum_i a_{ri}(\tau) p_{ri} z_{j(r,i)}$  on the preimage of  $(\mathrm{Aut}_\Lambda(\mathbf{P}))_u.C$  under  $\phi$ . Hence, the claim holds in this case.

Now let  $Q$  be a direct summand of  $\mathbf{P}$ . Without loss of generality, we may assume that the distinguished top elements  $z_1, \dots, z_m$  of  $Q$  coincide with the first  $m$  candidates on the distinguished list  $z_1, \dots, z_d$  for  $\mathbf{P}$ . In particular,  $\mathbf{P} = Q \oplus Q_1$ , where  $Q_1 = \bigoplus_{m+1 \leq r \leq d} \Lambda z_r$ . As explained in the remarks preceding Definition 2.7, we may identify the projective variety  $[\mathrm{GRASS}_d(\Lambda)]_Q$  with an isomorphic copy inside  $\mathrm{GRASS}_d(\Lambda)$ ; namely, with the closed subvariety of  $\mathrm{GRASS}_d(\Lambda)$  consisting of those points  $C$  which contain  $Q_1$ . In other words, we identify the points in  $[\mathrm{GRASS}_d(\Lambda)]_Q$  with those points in  $\mathrm{GRASS}_d(\Lambda)$  which have the form  $C = D \oplus Q_1$ . That  $M'$  be a unipotent degeneration of  $M = Q/C$  with  $C \in [\mathrm{GRASS}_d(\Lambda)]_Q$  such that  $M'/JM' \leq Q/JQ$ , means that there exists a point  $C'$  in the intersection of  $[\mathrm{GRASS}_d(\Lambda)]_Q$  with the closure of  $(\mathrm{Aut}_\Lambda(\mathbf{P}))_u.C$  in  $\mathrm{GRASS}_d(\Lambda) = [\mathrm{GRASS}_d(\Lambda)]_{\mathbf{P}}$  such that  $M' \cong \mathbf{P}/C'$ . Write  $C' = D' \oplus Q_1$ . Due to projectivity of the intersection, we can find a curve

$$\psi : \mathbb{P}^1 \rightarrow [\mathrm{GRASS}_d(\Lambda)]_Q \cap \overline{(\mathrm{Aut}_\Lambda(\mathbf{P}))_u.C}$$

such that  $C' = \lim_{\tau \rightarrow \infty} \psi(\tau)$ . By the special case  $Q = \mathbf{P}$ , the restriction of this curve to the preimage of  $(\mathrm{Aut}_\Lambda(\mathbf{P}))_u.C$  under  $\psi$  has the form  $U : \tau \mapsto h_\tau(C)$ , for some dense open subset  $U \subseteq K$  and an automorphism  $h_\tau$  given by  $h_\tau(z_r) = z_r + \sum_i a_{ri}(\tau) p_{ri} z_{j(r,i)}$  for  $1 \leq r \leq d$ ,  $1 \leq j(r,i) \leq d$ , suitable paths  $p_{ri} z_{j(r,i)}$  in  $J\mathbf{P}$  ending in  $e(r)$ , respectively,



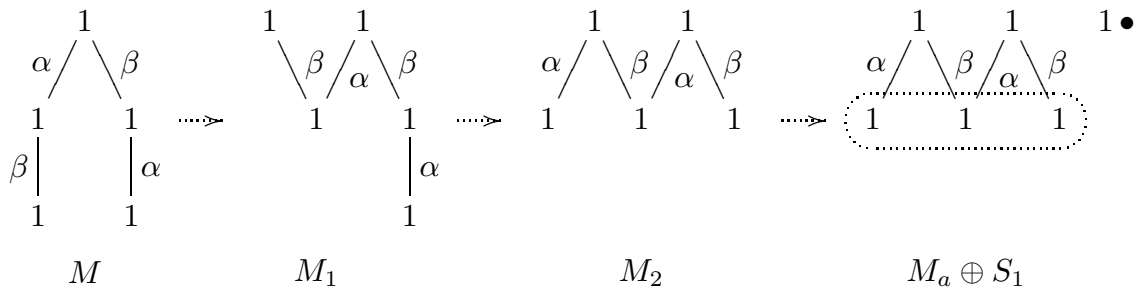
and suitable  $a_{ri}(\tau) \in K(\tau)$ . We split each  $h_\tau$  into its components:  $g_\tau \in (\text{Aut}_\Lambda(Q))_u$  plus  $h_\tau^{(1)} \in (\text{Aut}_\Lambda(Q_1))_u$  plus  $h_\tau^{(2)} \in \text{Hom}_\Lambda(Q, JQ_1)$  plus  $h_\tau^{(3)} \in \text{Hom}_\Lambda(Q_1, JQ)$ . By the choice of  $\psi$ , we have  $Q_1 \subseteq h_\tau(C)$ . We infer  $Q_1 \subseteq h_\tau(Q_1)$ , and hence  $Q_1 = h_\tau(Q_1)$  for reasons of dimension. Consequently,  $h_\tau^{(3)} = 0$ . Moreover, we do not alter the image  $h_\tau(C)$  by taking  $h_\tau^{(1)}$  to be the identity on  $Q_1$  and  $h_\tau^{(2)}$  to be zero. It is therefore harmless to assume that  $h_\tau = g_\tau \oplus \text{id}_{Q_1}$ , which implies  $\lim_{\tau \rightarrow \infty} g_\tau(D) = D'$  for the unipotent automorphisms  $g_\tau$  in  $\text{Aut}_\Lambda(Q)$ . This completes the argument.  $\square$

**Corollary 4.5.** *Suppose that  $M \cong P/C$  with  $C \in \mathbf{Grass}_d^T$  and that  $M'$  is a top-stable unipotent degeneration of  $M$ . Then  $M' = P/C'$ , where  $C'$  belongs to the closure of the orbit  $(\text{Aut}_\Lambda(P))_u.C$  in  $\mathbf{Grass}_d^T$ .  $\square$*

We close with an example to illustrate how Proposition 4.4 can be applied to finding unipotent degenerations.

**Example 4.6.** Let  $\Gamma$  be the quiver with a single vertex labeled 1 and two loops,  $\alpha$  and  $\beta$ . We consider the special biserial algebra  $\Lambda = K\Gamma/I$ , where  $I \subseteq K\Gamma$  is the ideal generated by  $\alpha^2$ ,  $\beta^2$ , and all paths of length 3. We denote the regular left  $\Lambda$ -module by  $\Lambda e_1$  to emphasize the side. Let  $M = \Lambda e_1$  and  $Q = \bigoplus_{1 \leq r \leq 3} \Lambda z_r \cong (\Lambda e_1)^3$ . We will discuss the unipotent degenerations of  $M$  with top  $S_1^3$ . For this purpose, we alternatively present  $M$  in the form  $Q/C$  with  $C = \Lambda z_1 \oplus \Lambda z_3$  in  $[\text{GRASS}_5(\Lambda)]_Q$ .

For  $a \in K^*$ , we denote by  $M_a$  the band module on the word  $\alpha\beta^{-1}\alpha\beta^{-1}$  corresponding to the  $2 \times 2$  Jordan matrix with eigenvalue  $a$ ; namely,  $M_a = (\Lambda z_1 \oplus \Lambda z_2)/U$ , where  $U$  is generated by  $\beta z_1 - \alpha z_2$  and  $\beta z_2 - a^2 \alpha z_1 + 2a \alpha z_2$ . We first show that, for any choice of  $a$ , the direct sum  $M_a \oplus S_1$  arises as a unipotent degeneration of  $M$ . In fact, the following is a sequence of successive unipotent degenerations:



Concerning the first link in the chain:  $M_1 = Q/C_1$  with  $C_1 = \lim_{\tau \rightarrow \infty} g_\tau.C$ , where, for  $\tau \in K$ , the map  $g_\tau \in (\text{Aut}_\Lambda(Q))_u$  is defined by  $z_1 \mapsto z_1 + \tau(\beta z_1 - \alpha z_2)$  and  $z_r \mapsto z_r$  for  $r = 2, 3$ . Indeed, from  $z_1 \in C$ , we first obtain  $g_\tau(\alpha\beta z_1) = \alpha\beta z_1 \in g_\tau.C$  and  $g_\tau(\alpha z_1) = (\alpha + \tau\alpha\beta)z_1 \in g_\tau.C$ , whence  $\alpha z_1 \in g_\tau.C$  for all  $\tau$ ; this shows  $\alpha z_1 \in C_1$ . Then, using the computation technique for 1-dimensional subspaces of  $\lim_{\tau \rightarrow \infty} g_\tau.C$  described in [9], Lemmas 4.7 and 5.5, we find

$$\lim_{\tau \rightarrow \infty} g_\tau(Kz_1) = \lim_{\tau \rightarrow \infty} K\left(\frac{1}{\tau}z_1 + (\beta z_1 - \alpha z_2)\right) = K(\beta z_1 - \alpha z_2) \subseteq C_1.$$

Thus,  $C_1 \supseteq \Lambda\alpha z_1 + \Lambda(\beta z_1 - \alpha z_2) + \Lambda z_3$ . Since the latter sum is 10-dimensional, and  $Q/C_1$  is 5-dimensional, we infer equality. This shows  $M_1$  to be a unipotent degeneration of  $M$ .

To obtain  $M_2$  as a unipotent degeneration of  $M_1$ , we consider  $g_\tau \in (\text{Aut}_\Lambda(Q))_u$  for  $\tau \in K$ , defined by  $z_1 \mapsto z_1 + \tau\beta z_2$ , and  $g_\tau(z_r) = z_r$  for  $r = 2, 3$ . Moreover, we set  $C_2 = \lim_{\tau \rightarrow \infty} g_\tau.C_1$ . From  $\alpha z_1 \in C_1$ , we infer  $\alpha z_1 + \tau\alpha\beta z_2 \in g_\tau.C_1$ , and hence, as above,

$$\lim_{\tau \rightarrow \infty} g_\tau(K\alpha z_1) = \lim_{\tau \rightarrow \infty} K\left(\frac{1}{\tau}\alpha z_1 + \alpha\beta z_2\right) = K\alpha\beta z_2 \subseteq C_2.$$

We next observe that  $\beta z_1 - \alpha z_2 \in C_1$  implies  $\beta z_1 - \alpha z_2 = g_\tau(\beta z_1 - \alpha z_2) \in g_\tau.C_1$ ; moreover,  $\alpha z_1 \in C_1$  implies  $\beta\alpha z_1 = g_\tau(\beta\alpha z_1) \in g_\tau.C_1$  for all  $\tau$ . Consequently, these containments also hold in the limit, which shows

$$C_2 \supseteq \Lambda\alpha\beta z_2 + \Lambda(\beta z_1 - \alpha z_2) + \Lambda\beta\alpha z_1 + \Lambda z_3.$$

A comparison of dimensions again gives equality.

To see that, for any  $a \in K^*$ , the module  $M_2$  unipotently degenerates to  $M_a \oplus S_1$ , we take  $g_\tau(z_r) = z_r$  for  $r = 1, 2$  and  $g_\tau(z_3) = z_3 + \tau(\beta z_2 - a^2\alpha z_1 + 2a\alpha z_2)$ . Due to  $z_3 \in C_2$ , this yields

$$\lim_{\tau \rightarrow \infty} g_\tau.C_2 = \Lambda\beta\alpha z_1 + \Lambda(\beta z_1 - \alpha z_2) + \Lambda(\beta z_2 - a^2\alpha z_1 + 2a\alpha z_2) + \Lambda\alpha z_3 + \Lambda\beta z_3,$$

as required.

Next we consider the  $\mathbb{P}^1$ -family of modules  $M_{[c_1:c_2]}$ , which, for  $c_1, c_2 \in K^*$ , can be visualized in the form

$$\begin{array}{c} 1 \\ \alpha \backslash \quad / \beta \\ 1 \end{array} \oplus \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \beta$$

While the modules  $M_{[1:0]}$  and  $M_{[0:1]}$ , each a direct sum of the displayed 3-dimensional string module and a string module of dimension 2, are unipotent degenerations of  $M$ , we believe the remaining candidates not to be.

All other 5-dimensional left  $\Lambda$ -modules with 3-dimensional top do arise as unipotent degenerations of  $M$ . There are six sporadic ones, the  $\mathbb{P}^1$ -family which we discussed at the outset (next to the band modules  $M_a \oplus S_1$  for  $a \in K^*$ , it contains two string modules), and the following  $\mathbb{P}^1 \times \mathbb{P}^1$ -family:

$$(\dagger) \quad (\Lambda/\Lambda(a_1\alpha - a_2\beta)) \oplus (\Lambda/\Lambda(b_1\alpha - b_2\beta)) \oplus S_1, \quad ([a_1:a_2], [b_1:b_2] \in \mathbb{P}^1).$$

To deal with the latter, consider the family  $(g_\tau)_{\tau \in K}$  in  $(\text{Aut}_\Lambda(Q))_u$  with  $g_\tau$  given by  $z_1 \mapsto z_1 + \tau(a_1\alpha z_2 - a_2\beta z_2)$ ,  $z_2 \mapsto z_2$ , and  $z_3 \mapsto z_3 + \tau(b_1\alpha z_1 - b_2\beta z_1)$ . Set  $C' = \lim_{\tau \rightarrow \infty} g_\tau.C$ . As above, one obtains  $a_1\alpha z_2 - a_2\beta z_2 \in C'$  and  $b_1\alpha z_1 - b_2\beta z_1 \in C'$  from  $z_1, z_3 \in C$ . Moreover,  $\alpha z_3 \in C'$ ; indeed,  $z_3 \in C$  guarantees that  $\alpha z_3 - \tau b_2\alpha\beta z_1 \in g_\tau.C$  for all  $\tau$ . From  $z_1 \in C$ ,

we additionally derive  $\alpha\beta z_1 \in g_\tau.C$  for all  $\tau$ ; this yields  $\alpha z_3 \in g_\tau.C$  for all  $\tau$ , and hence  $\alpha z_3 \in C'$ . Analogously, one shows that  $\beta z_3 \in C'$ . Consequently,

$$C' \supseteq \Lambda(a_1\alpha z_2 - a_2\beta z_2) + \Lambda(b_1\alpha z_1 - b_2\beta z_1) + \Lambda\alpha z_3 + \Lambda\beta z_3.$$

Since the right hand module is 10-dimensional and  $\dim Q/C' = 5$ , we again derive equality, that is,  $Q/C'$  equals a typical member of the family  $(\dagger)$ .

Finally, we mention that, in particular, the modules  $N_1$  and  $N_2$  with graphs

$$\begin{array}{c} 1 & & 1 & & 1 \\ & \backslash & / & \backslash & / \\ \alpha & & \beta & & \alpha \\ & / & \backslash & / & \backslash \\ 1 & & 1 & & 1 \end{array} \quad \text{and} \quad \begin{array}{c} 1 & & 1 & & 1 \bullet \\ & \backslash & / & \backslash & / \\ \alpha & & \beta & & \alpha \\ & / & \backslash & / & \backslash \\ 1 & & 1 & & 1 \end{array}$$

are unipotent degenerations of  $M$ . However, while  $N_1$  degenerates to  $N_2$ , this degeneration is not unipotent, by Theorem 4.3, since  $\mathbb{S}(N_1) \neq \mathbb{S}(N_2)$ .

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